Coulomb corrections in (e,e') using the eikonal expansion

Accuracy + insight

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contents:

Introduction

- **The goal is to measure nuclear response functions.**
- Use θ_e dependence to separate σ_L and σ_T?
- Electron distortion by Coulomb potential is a complication.
- **How to remove the Coulomb effects in a reliable way?**

What seems to be agreed upon?

- Remove radiation effects from experimental data.
- **Compare with DWIA using Dirac-Coulomb waves for the electron.**
- Use theoretical analysis as a guide to analysis of data.

The issues.

- Approximate methods that provide insight do not provide accuracy.
- Exact methods that provide accuracy do not provide insight.
- Nontrivial numerics:
  - Slowly converging partial wave expansions.
  - Multi-dimensional integrations.
- Different results from different groups.

What to do?

- **Better approximations - systematic and accurate ones.**
- **Agreement between different theoretical methods.**
  - Insight and accuracy is the goal.

Cross checks of “black-box” results.

- **Analytical electron wave functions**

K-G Eikonal approximation

Klein-Gordon equation

$$([E - V(r)]^2 - \mathbf{p}^2 - m^2)\psi(\mathbf{r}) = 0$$

$$\psi_{\mathbf{k}}^{(+)}(\mathbf{r}) \approx e^{ikz} \;\; e^{i\chi_0^{(+)}(\mathbf{r})},$$

$$\chi_0^{(+)}(\mathbf{r}) = -\frac{1}{v}\int_{-\infty}^{z} dz' V(r')$$

$$r' = \sqrt{z'^2 + b^2}$$

- How accurate?
- What about the "focusing factor"?

K-G Eikonal expansion

$$\psi_{\mathbf{k}}(\mathbf{r}) = e^{ikz}\, e^{i\chi^{(+)}}\, e^{-\omega^{(+)}},$$

$$\begin{aligned} \chi^{(+)} &= \chi_0^{(+)} + \chi_1^{(+)} + \chi_2^{(+)} + \cdots \\ \omega^{(+)} &= \qquad\quad \omega_1^{(+)} + \omega_2^{(+)} + \cdots . \\ & \sim 1 \quad \sim \frac{V}{E} \quad \sim \frac{V^2}{E^2} \quad \sim \frac{V^3}{E^3} \end{aligned}$$

- Asymptotic series.
- $V/E \approx$0.05 for ^{208}Pb, E=500 MeV.
- Error $\approx$ first term omitted $\approx$ .00125.

Leading corrections to eikonal

$$\chi_{\mathbf{1}}^{(+)}(\mathbf{r}) = -\frac{1}{2k}\int_{-\infty}^{z} dz' \Big([\nabla' \chi_0^{(+)}(\mathbf{r}')]^2 - V^2(r') \Big)$$

$$\omega_1^{(+)}(\mathbf{r}) = \frac{1}{2k}\int_{-\infty}^{z} dz' \nabla'^2 \chi_0^{(+)}(\mathbf{r}')$$

Analytical results for $V(r) = -Z\alpha/\sqrt{r^2+R^2}$

Let $u = \sqrt{r^2+R^2}$ **and** $w = \sqrt{b^2+R^2}$**, then**

$$\chi_{\mathbf{0}}^{(+)}(\mathbf{r}) = \frac{\alpha Z}{v} ln\left(\frac{z+u}{w^2}\right)$$

$$\chi_1^{(+)}(\mathbf{r}) = -\frac{(Z\alpha)^2 b^2}{k v^2 w^4}\left(z + u + \frac{1}{2} w tan^{-1}\left(\frac{w}{z}\right)\right)$$

$$\omega_1^{(+)}(\mathbf{r}) = -\frac{Z\alpha R^2}{k v w^4}\left(z + u - \frac{w^2}{2u}\right)$$

K-G focusing factor

$$f_{KG}(\mathbf{r}) = e^{-\omega^{(+)}(\mathbf{r})}$$

- **Enhanced ψ for e$^-$ wave near nucleus.**

$$\omega^{(+)}(\mathbf{0}) = \frac{V(0)}{2E}$$

$$f_{KG}(\mathbf{0}) \approx 1 - \frac{V(0)}{2E}$$

- **Eikonal expansion required to get $f_{KG} \neq 1$.**

- **Dirac-Coulomb wave has different focusing factor.**

Dirac-Coulomb Eikonal expansion

$$\psi(\mathbf{r}) = \begin{pmatrix} u(\mathbf{r}) \\ \ell(\mathbf{r}) \end{pmatrix}$$

Solve for upper-component spinor:

$$\left(E_1 - V - \sigma \cdot \mathbf{p} \frac{1}{E_2 - V} \sigma \cdot \mathbf{p}\right) u(\mathbf{r}).$$

$E_1 = E - m \approx E$.

Then get lower component spinor:

$$\ell(\mathbf{r}) = \frac{1}{E_2 - V}\, \sigma \cdot \mathbf{p}\ u(\mathbf{r}).$$

$E_2 = E + m \approx E$.

Upper component solution

$$\mathbf{u}_\lambda^{(+)}(\mathbf{r}) = \mathbf{f_D}(\mathbf{r}) \quad \mathbf{e}^{\mathbf{ikz}} \quad \mathbf{e}^{\mathbf{i}\chi^{(+)}(\mathbf{r})} \quad \mathbf{e}^{\mathbf{i}\sigma_\mathbf{e}\bar{\gamma}^{(+)}(\mathbf{r})} \quad \xi_\lambda$$

$$\sigma_\mathbf{e} = \sigma \cdot \hat{\mathbf{b}} \times \hat{\mathbf{z}}$$

$$\mathbf{f_D}(\mathbf{r}) = \left(\mathbf{1} - \frac{\mathbf{V}}{\mathbf{E_2}}\right)^{\mathbf{1/2}} \mathbf{e}^{-\omega^{(+)}(\mathbf{r})}$$

- $\chi^{(+)} = \chi_0^{(+)} + \chi_1^{(+)} + \chi_2^{(+)} + \cdots$
- $\omega^{(+)} = \omega_1^{(+)} + \omega_2^{(+)} + \cdots$
- **Same** $\chi_0^{(+)}, \chi_1^{(+)}, \omega_1^{(+)}$ **as for K-G.**
- $\bar{\gamma}^{(+)} = \frac{1}{2E}\left(\frac{\partial \chi^{(+)}}{\partial b} + i\frac{\partial \omega^{(+)}}{\partial b}\right)$.

Lower component for $m_e = 0$

$$\begin{aligned} \ell_\lambda(\mathbf{r}) &= 2\lambda u_\lambda(\mathbf{r}) \\ \lambda &= \pm\frac{1}{2}. \end{aligned}$$

Proof

$$\begin{aligned} \tilde{u}_\lambda &= \sqrt{E-V}\,u_\lambda \\ \tilde{\ell}_\lambda &= \sqrt{E-V}\,\ell_\lambda \\ h &= \frac{1}{\sqrt{E-V}}\sigma\cdot\mathbf{p}\frac{1}{\sqrt{E-V}} \end{aligned}$$

$$\begin{aligned} \tilde{u}_\lambda(\mathbf{r}) &= h\tilde{\ell}_\lambda \qquad \tilde{\ell}_\lambda(\mathbf{r}) = h\tilde{u}_\lambda(\mathbf{r}) \\ \rightarrow \quad h^2\tilde{u}_\lambda &= \tilde{u}_\lambda \end{aligned}$$

Eigenvalues: $h = 2\lambda = \pm 1$.

Error $\approx 10^{-5}$.

Current matrix element

$$J_{fi}^{\mu} = \int d^3r \bar{\Psi}_{\mathbf{k}_f\lambda_f}^{(-)}(\mathbf{r})\gamma^{\mu}\Psi_{\mathbf{k}_i\lambda_i}^{(+)}(\mathbf{r}),$$

$$\Psi_{\mathbf{k}_i\lambda_i}^{(+)}(\mathbf{r}) = \frac{1}{\sqrt{2}}\begin{pmatrix} u_{\lambda_i}^{(+)}(\mathbf{r}) \\ 2\lambda_i\; u_{\lambda_i}^{(+)}(\mathbf{r}) \end{pmatrix},$$

Helicity is conserved.

$$J_{fi}^{\mu} = \delta_{\lambda_f\lambda_i}\int d^3r e^{i(\mathbf{Q}-\mathbf{q})\cdot\mathbf{r}} f_i(\mathbf{r}) f_f(\mathbf{r}) e^{i\chi(\mathbf{r})} j_e^{\mu}(\mathbf{r})$$

$$\chi(\mathbf{r}) = \chi_f^{(-)}(\mathbf{r}) + \chi^{(+)}(\mathbf{r})$$

$$f_i(\mathbf{0}) f_f(\mathbf{0}) \approx \left(1 - \frac{V(0)}{\epsilon_f}\right)\left(1 - \frac{V(0)}{\epsilon_i}\right)$$

Helicity matrix elements

θ_e is electron scattering angle.

$$
\begin{aligned}
j_e^0 &= cos\frac{1}{2}\theta_e + sin\frac{1}{2}\theta_e\Big(X_i + X_f\Big) \\
j_e^{T_1} &= sin\frac{1}{2}\theta_e\frac{k_i + k_f}{|\mathbf{Q}|} + cos\frac{1}{2}\theta_e\frac{\omega}{|\mathbf{Q}|}\Big(X_i - X_f\Big) \\
j_e^{T_2} &= \Big(2i\lambda_i\Big)\Big[sin\frac{1}{2}\theta_e - cos\frac{1}{2}\theta_e\Big(X_i + X_f\Big)\Big] \\
j_e^L &= cos\frac{1}{2}\theta_e\frac{\omega}{|\mathbf{Q}|} - sin\frac{1}{2}\theta_e\frac{k_i + k_f}{|\mathbf{Q}|}\Big(X_i - X_f\Big)
\end{aligned}
$$

$$X_i = sin\gamma_i^{(+)}e^{2i\lambda_i\phi_i} \sim \frac{V}{\epsilon_i},$$

$$X_f = sin\gamma_f^{(-)}e^{-2i\lambda_f\phi_f} \sim \frac{V}{\epsilon_f}.$$

- **plane wave θ_e dependence in blue.**
- **New helicity dependent terms in red.**

Focusing factors & currents

Dirac wave function

$$f_i(\mathbf{0})f_f(\mathbf{0}) \approx \left(1 - \frac{V(0)}{E}\right)\left(1 - \frac{V(0)}{E}\right)$$

Dirac current j_e^0

$$cos\frac{1}{2}\theta_e + sin\frac{1}{2}\theta_e\left(X_i + X_f\right)$$

Klein-Gordon wave function

$$f_i(\mathbf{0})f_f(\mathbf{0}) \approx \left(1 - \frac{V(0)}{E}\right)$$

Klein-Gordon current $j_e^\mu(\mathbf{r})$

$$\left[\epsilon_i + \epsilon_f - 2V(0), \mathbf{k}_i + \nabla\chi_i^{(+)} + \mathbf{k}_f - \nabla\chi_f^{(-)})\right]$$

$$\approx \left(1 - \frac{V(0)}{E}\right)[\epsilon_i + \epsilon_f, \mathbf{k}_i + \mathbf{k}_f]$$

DWIA matrix element

$$\frac{d\sigma}{d\Omega_f d\epsilon_f} = \int d\Omega_p \frac{4\alpha^2}{(2\pi)^5} |\mathcal{M}|^2 \epsilon_f^2 p E_p$$

$$\mathcal{M} = \int d^3r \int \frac{d^3q}{(2\pi)^3} e^{i(\mathbf{Q}-\mathbf{q})\cdot\mathbf{r}} e^{i\chi(\mathbf{r})} \times$$
$$f_f(\mathbf{r}) f_i(\mathbf{r}) j_e^{\mu}(\mathbf{r}) \left(\frac{1}{\mathbf{q}^2 - \omega^2} \right) J_{\mu}^{N}(\mathbf{q}, \mathbf{p})$$

Effective momentum approximation:

$$\mathbf{q} \approx \mathbf{Q}_{eff} = \mathbf{Q} + \nabla\chi(\mathbf{0})$$

$$\mathcal{M}_{eff} = \left(\frac{1}{\mathbf{Q}_{eff}^2 - \omega^2} \right) \int d^3r e^{i\mathbf{Q}\cdot\mathbf{r}} e^{i\chi(\mathbf{r})} \times$$
$$f_f(\mathbf{r}) f_i(\mathbf{r}) j_e^{\mu}(\mathbf{r}) \widetilde{J}_{\mu}^{N}(\mathbf{r}, \mathbf{p})$$

Correction

Use identity

$$\frac{1}{\mathbf{q}^2 - \omega^2} = \frac{1}{\mathbf{Q}_{eff}^2 - \omega^2} + \frac{1}{\mathbf{Q}_{eff}^2 - \omega^2}\left(\mathbf{Q}_{eff}^2 - \mathbf{q}^2\right)\frac{1}{\mathbf{q}^2 - \omega^2}$$

$$\mathcal{M} = \mathcal{M}_{eff} + \delta\mathcal{M}$$

$$\delta\mathcal{M} = \int d^3r \int \frac{d^3q}{(2\pi)^3} e^{i(\mathbf{Q}-\mathbf{q})\cdot\mathbf{r}} e^{i\chi(\mathbf{r})} f_f(\mathbf{r}) f_i(\mathbf{r}) j_e^{\mu}(\mathbf{r}) \times \left[\frac{1}{\mathbf{Q}_{eff}^2 - \omega^2}\left(\mathbf{Q}_{eff}^2 - \mathbf{q}^2\right)\frac{1}{\mathbf{q}^2 - \omega^2}\right] J_{\mu}^{N}(\mathbf{q}, \mathbf{p})$$

- **6D integral for correction not done.**

Calculations

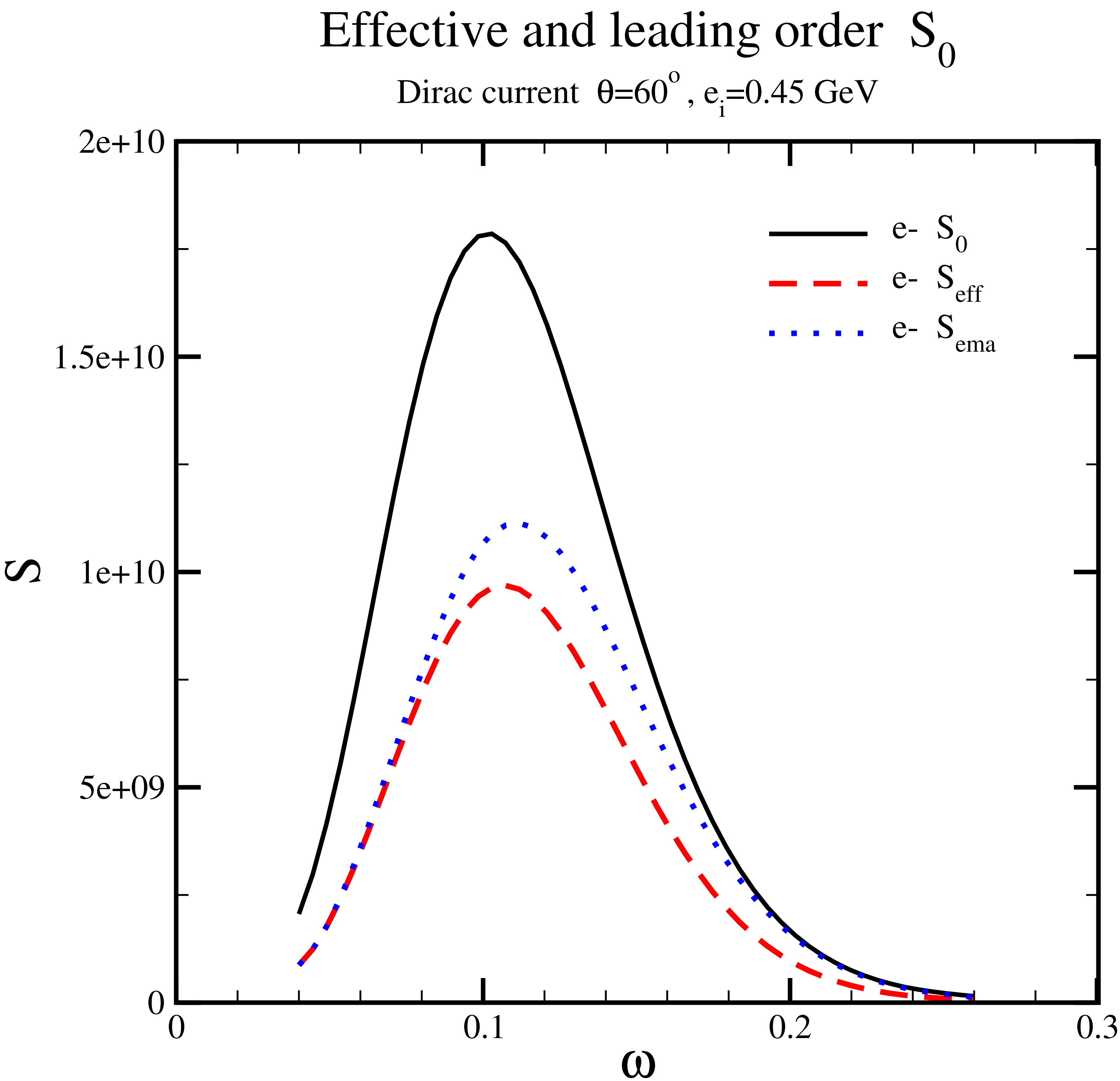
$$\frac{f_i(\mathbf{0})f_f(\mathbf{0})}{\mathbf{Q}^2_{eff} - \omega^2} = \frac{1}{\mathbf{Q}^2 - \omega^2} \quad (Traini)$$

- **Leads to overall factor σ_{Mott}.**
- **What is the effect of the new terms in j_e^μ from spin-dependence of eikonal?**

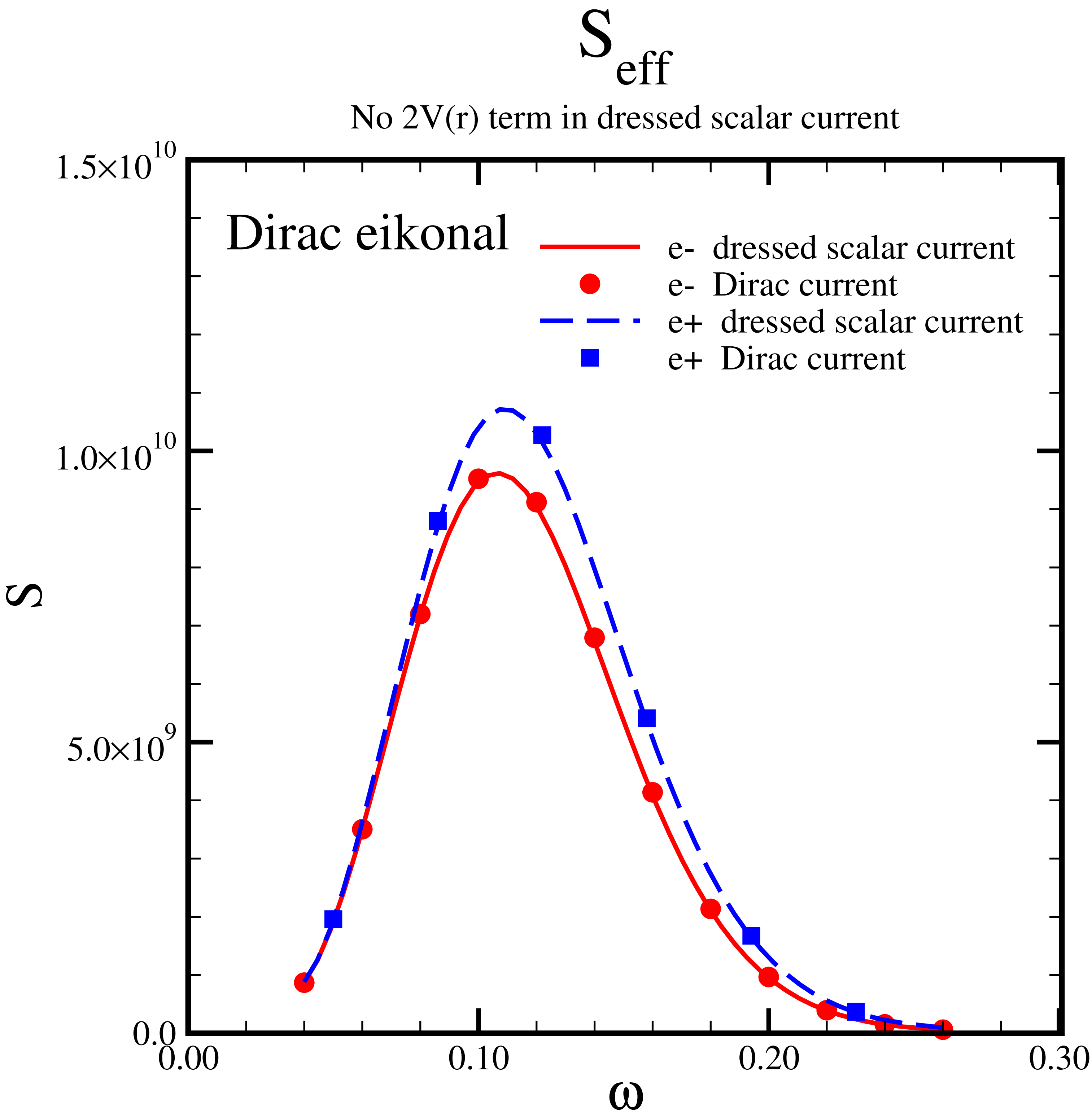
To get an idea we calculate response function $\mathbf{S}_{eff}$

$$\mathcal{S}_{eff} = \frac{1}{\sigma_{Mott}} \frac{d\sigma_{eff}}{d\Omega_f d\epsilon_f}$$

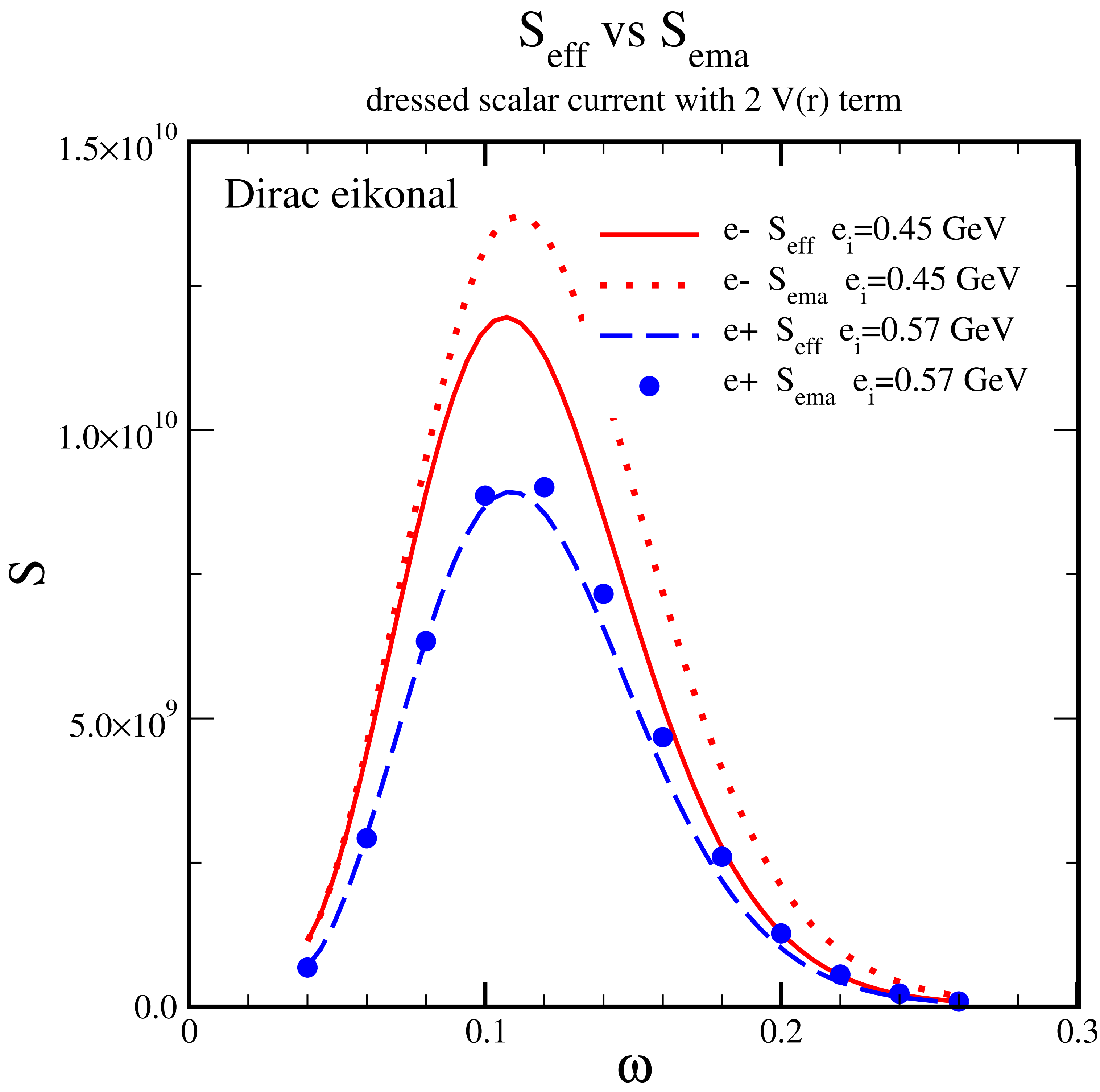
- eff **denotes inclusion of new terms in j_e^μ, which mainly affect the $j_e^L J_N^L$ contribution.**
- ema **denotes omission of new terms.**



—— S_0 **uses** Q^2; – – – S_{eff} **uses** Q^2_{eff}; $\cdots S_{ema}$ **uses** Q^2_{eff} **but omits longitudinal current** $J_e^L J_N^L$.

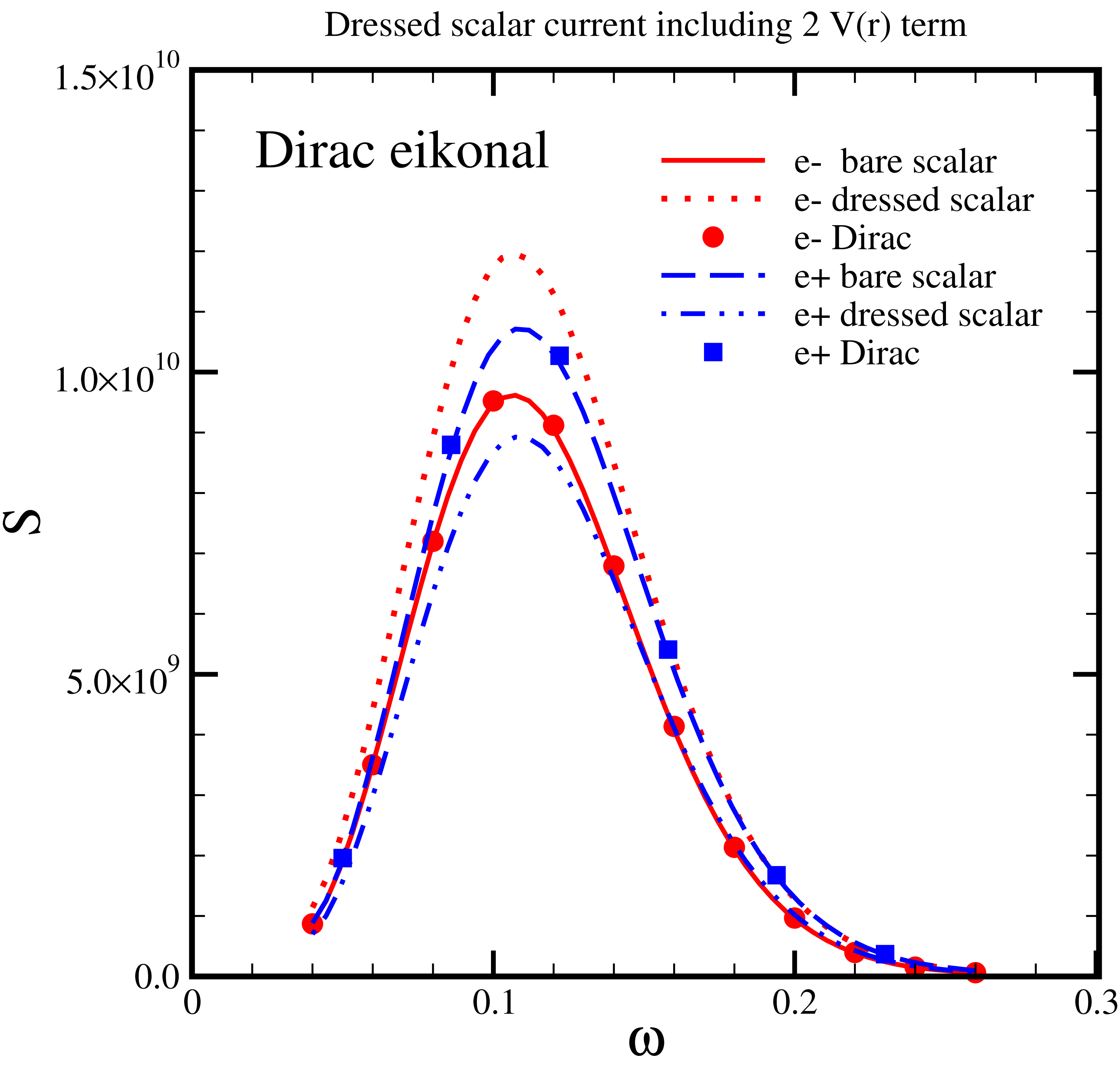


Comparison of currents: Klein-Gordon e^-(——) and e^+ (– – –); filled circles and squares show Dirac current.



Comparison of currents: Klein-Gordon e^-(———) and e^+ (— — —); filled circles and squares show Dirac current.

S_{eff}



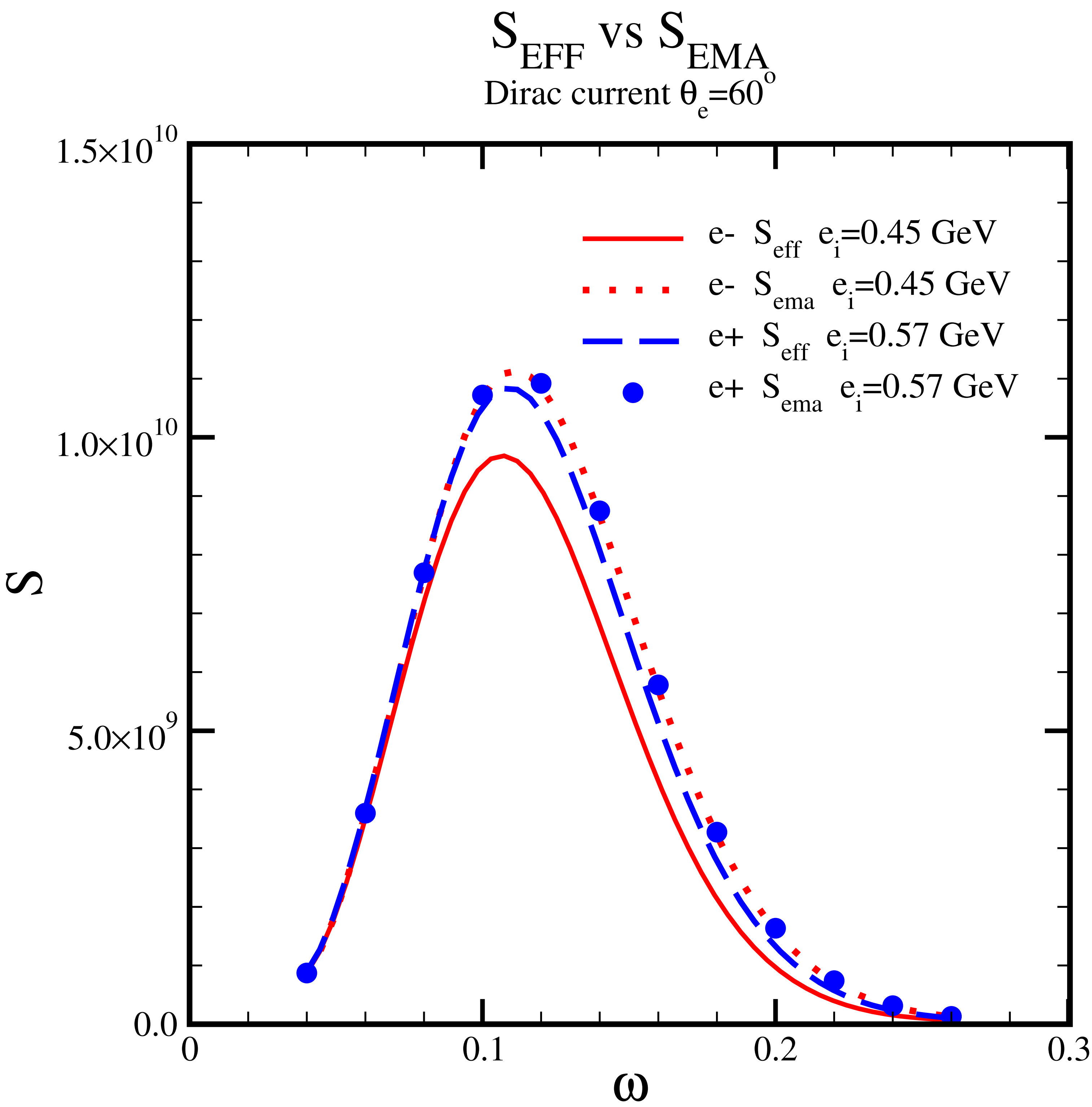
Comparisons of e^- and e^+ using Dirac eikonal. The use of either Dirac current or bare scalar current gives equivalents results.

- **OK to use of Dirac focusing factor with bare scalar current.**

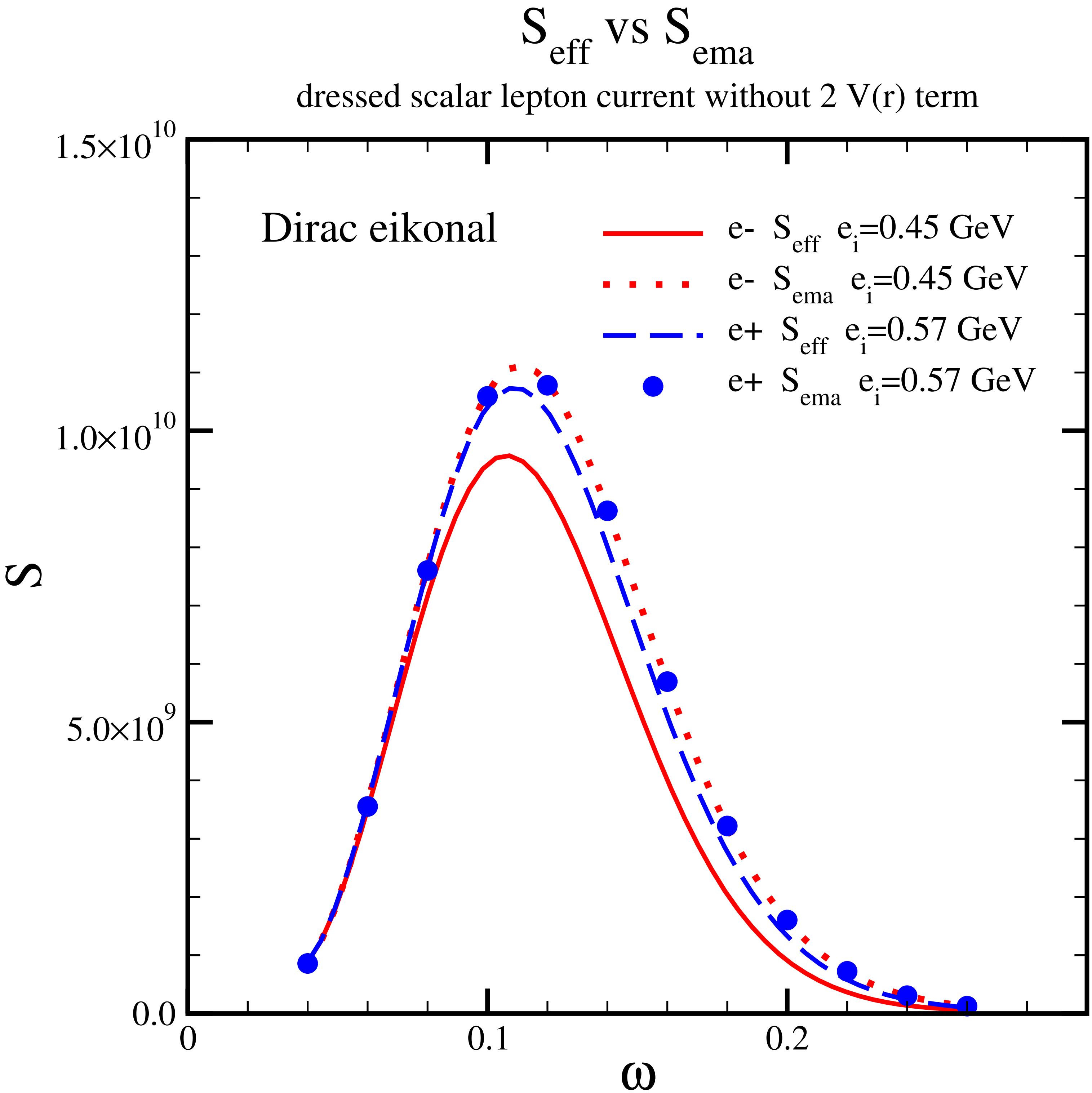
- **OK to use KG focusing factor with dressed KG current.**

- **Use of Dirac focusing factors with dressed scalar current is not consistent.**

- **Three factors of** $\left(1 - \frac{V(0)}{E}\right)$ **is one too many.**



Dirac current: Comparison of e^- (red) and e^+ (blue) for ema (omits $j_e^L J_N^L$ term) and eff (includes $j_e^L J_N^L$ term)



Scalar current: Comparison of e^- (red) and e^+ (blue) for ema (omits $j_e^L J_N^L$ term) and eff (includes $j_e^L J_N^L$ term)

Summary

- **Eikonal expansion provides an accurate way to determine the DWIA wave functions.**

- **Analytical results for** $V(r) = \frac{Z\alpha}{\sqrt{r^2+R^2}}$**.**

- **Lower component** $\ell_{\pm\frac{1}{2}}(\mathbf{r}) = \pm \;\; u_{\pm\frac{1}{2}}(\mathbf{r})$**.**

- **Nontrivial corrections to current matrix elements.**

- **Response functions for** e^+ **and** e^- **very close based on** $j_e^0 J_N^0$ **coupling. (**ema**)**

- **Calculations for** e^+ **and** e^- **differ at 10-15% level when** $j_e^L J_N^L$ **term is included. (**eff**)**