Lattice QCD at non-zero temperature and density

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- QCD in a nutshell, non-perturbative physics, lattice-regularized QCD, Monte Carlo simulations
- the phase diagram on strongly interacting matter, chiral symmetry restoration, the equation of state
- finite density QCD, cumulants of conserved charge fluctuations, thermal masses & transport properties

Discretization of fermion fields

$$\mathcal{L}_E = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar{\psi}_{j,a} \left(\sum_{\nu=0}^{3} \gamma_\nu \left(\partial_\nu - i \frac{g}{2} \mathcal{A}^a_\nu \lambda^a \right) + m_j \right)^{a,b} \psi_{j,b}$$

$$\psi(x) \rightarrow \psi_n \quad , \quad \bar{\psi}(x) \rightarrow \bar{\psi}_n$$

discretization of first order derivative in fermionic part is straightforward:

$$\bar{\psi}(x) \partial_\mu \psi(x) \quad \rightarrow \quad \bar{\psi}_n \left(\frac{\psi_{n+\mu} - \psi_{n-\mu}}{2a} \right)$$

$$\bar{\psi}_n \psi_{n+\mu} - \bar{\psi}_n \psi_{n-\mu}$$

discretization of derivative generates
point-split terms $\Rightarrow$ local gauge invariance?

gauge transformation: $\bar{\psi}_n \psi_{n+\mu} \;\Rightarrow\; \bar{\psi}_n G_n^{-1} G_{n+\mu} \psi_{n+\mu}$

transformation of parallel transporter: $U_\mu(n) \;\rightarrow\; G_n U_\mu(n) G^{-1}_{n+\mu}$

$$\bar{\psi}(x) \gamma_\mu \left(\partial_\mu + i \frac{g}{2} \mathcal{A}^a_\mu \lambda^a \right) \psi(x) \quad \rightarrow \quad \bar{\psi}_n \gamma_\mu U_\mu(n) \psi_{n+\mu}$$

– symmetrize discretized derivative using forward and backward differences:

$$\bar\psi(x)\gamma_\mu\left(\partial_\mu + i\frac{g}{2}\mathcal{A}^a_\mu\lambda^a\right)\psi(x) \quad\rightarrow\quad \bar\psi_n\gamma_\mu U_\mu(n)\psi_{n+\mu} - \bar\psi_n\gamma_\mu U^\dagger_\mu(n-\mu)\psi_{n-\mu}$$

Fermion doubler:

$$S_F = -\int_0^t \mathrm{d}t' \int \mathrm{d}^3x\ \bar\psi(t',\vec{x})\left[\sum_{\mu=0}^{3}\gamma_\mu(\partial_\mu + igA_\mu(t',\vec{x})) + m\right]\psi(t',\vec{x})$$

lattice regularization $\psi(x) \rightarrow \psi_n \equiv \psi_n^{a,\nu,f}$

$a = 1, ..., N_c,$ color
$\nu = 1, ..., 4,$ spinor
$f = 1, .., n_f,$ flavor

$$S_F = \sum_{n,m}\bar\psi_n M_{nm}\psi_m$$

Grassmann variables:
$\psi_n\psi_m = -\psi_m\psi_n$
$\psi_n^2 = 0$

naïve discretization scheme

$$M[U]_{nm} \quad = \quad m\,\delta_{nm} + D[U]_{nm}$$

$$D[U]_{nm} = \frac{1}{2}\sum_{\mu=0}^{3}\gamma_\mu\left(U_\mu(n)\delta_{n,m-\mu} - U^\dagger_\mu(n-\mu)\delta_{n,m+\mu}\right)$$

The fermion doubler problem:

consider free fermions: $U_\mu(n) \equiv 1$

$$M_{nm} = m\,\delta_{nm} + D_{nm}$$

$$D_{nm} = \frac{1}{2}\sum_{\mu=0}^{3}\gamma_\mu\left(\delta_{n,m-\mu} - \delta_{n,m+\mu}\right)$$

introduce Grassmann fields in momentum space:

$$\boxed{\begin{gathered} p_i = \frac{2\pi}{N_\sigma} n_i \\ p_0 = \frac{2\pi}{N_\tau}(n_0 + 1/2) \\ 0 \le n_\mu \le N_\mu - 1 \end{gathered}}$$

$$\bar\psi_x = \frac{1}{\sqrt{N_\sigma^3 N_\tau}}\sum_p \mathrm{e}^{-ipx}\bar\psi_p \quad , \quad x \equiv (t, \vec{x})$$

$$\psi_x = \frac{1}{\sqrt{N_\sigma^3 N_\tau}}\sum_p \mathrm{e}^{ipx}\psi_p$$

$$\Rightarrow \quad S = \frac{1}{N_\sigma^3 N_\tau}\sum_x\sum_{p,p'} \mathrm{e}^{ix(p-p')}\bar\psi_{p'}^\alpha \left[\frac{1}{2}\sum_\mu \gamma_\mu(\mathrm{e}^{ip_\mu} - \mathrm{e}^{-ip_\mu}) + m\cdot 1\right]^{\alpha\beta} \psi_p^\beta$$

$$= \sum_p \bar\psi_p^\alpha \underbrace{\left[i\sum_\mu \gamma_\mu \sin p_\mu + m\cdot 1\right]^{\alpha\beta}}_{(M_p)_{\alpha\beta}} \psi_p^\beta$$

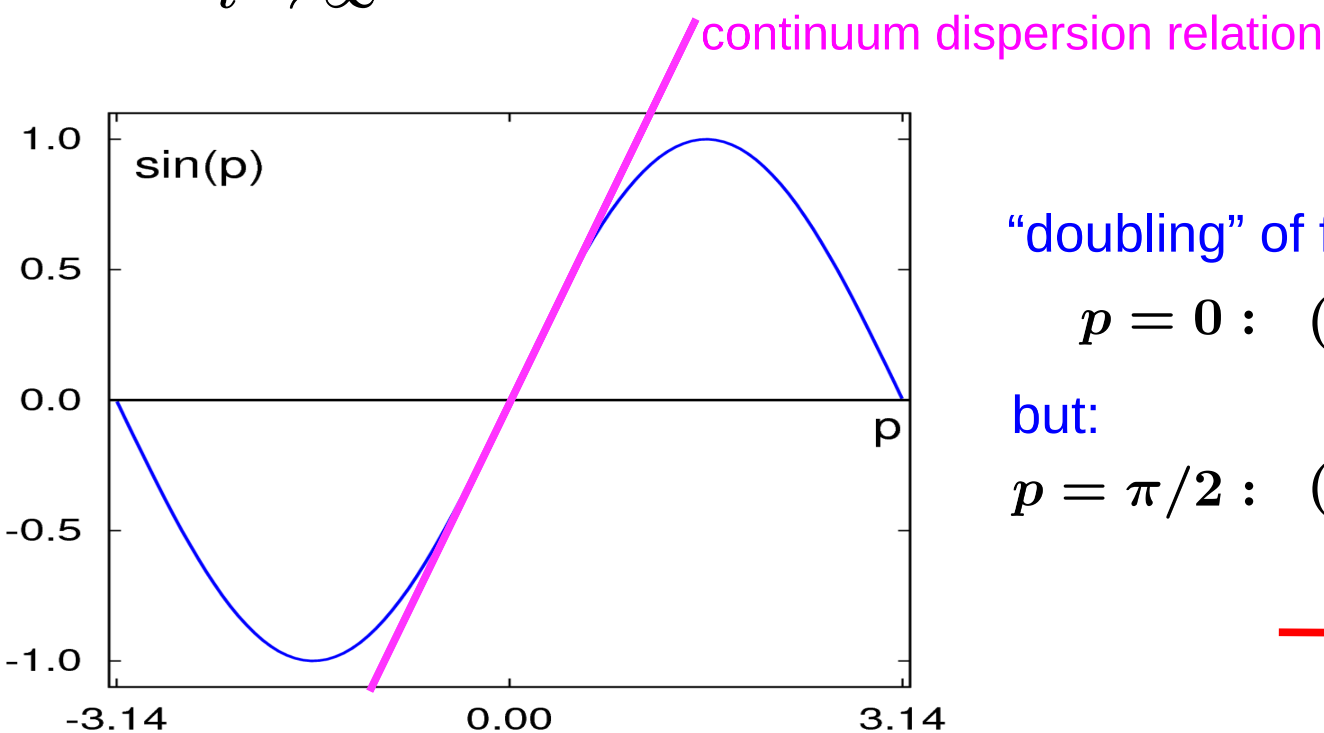
$$\langle \bar\psi_p^\alpha \psi_p^\beta \rangle = (M^{-1})^{\alpha\beta}$$

$(M_p)_{\alpha\beta}$ diagonal in momentum space consequences become apparent in 2-particle correlation function:

$$M_p^{-1} = (M_p^\dagger M_p)^{-1} M_p^\dagger = \frac{1}{m^2 + \sum_\mu \sin^2 p_\mu} M_p^\dagger$$

$$\begin{aligned} G(t,\vec{p}) &= \sum_{\vec{x}} \mathrm{e}^{i\vec{p}\vec{x}} \langle \bar{\psi}(t,\vec{x})^\alpha \psi(0,\vec{0})^\beta \rangle \\ &= \sum_{p_0} \mathrm{e}^{-ip_0 t} \frac{[m\cdot 1 - i\gamma_\mu \sin p_\mu]^{\alpha\beta}}{\sin^2 p_0 + \omega^2(\vec{p})} \quad , \quad \omega(\vec{p}) = \left(m^2 + \sum_{i=1}^{3} \sin^2 p_i \right)^{1/2} \\ &\underset{t\to\infty}{\sim} \mathrm{e}^{-tE(\vec{p})} \end{aligned}$$

16 poles for m=0 → 16 massless states



“doubling” of fermion modes

$$p = 0 : \quad (Ea)^2 = (pa)^2$$

but:

$$p = \pi/2 : \quad (Ea)^2 = 1 + (ma)^2$$

→ Wilson fermions

Wilson-fermions:

introduce heavy fermions with mass $ma \sim 1$ to remove doublers

$$m \sim 1/a \to \infty$$

will decouple in the continuum limit

$$S_{naive} = \sum_{n,m} \bar{\psi}_n \left[\frac{1}{2} \sum_{\mu=0}^{3} \gamma_\mu \left(\delta_{n,m-\mu} - \delta_{n,m+\mu} \right) + \hat{m}\delta_{n,m} \right] \psi_m$$

$$S_{Wilson} = S_{naive} - \sum_{n,m} \bar{\psi}_n \left[\frac{1}{2} \sum_{\mu=0}^{3} \left(\delta_{n,m-\mu} + \delta_{n,m+\mu} \right) - 4\delta_{n,m} \right] \psi_m$$

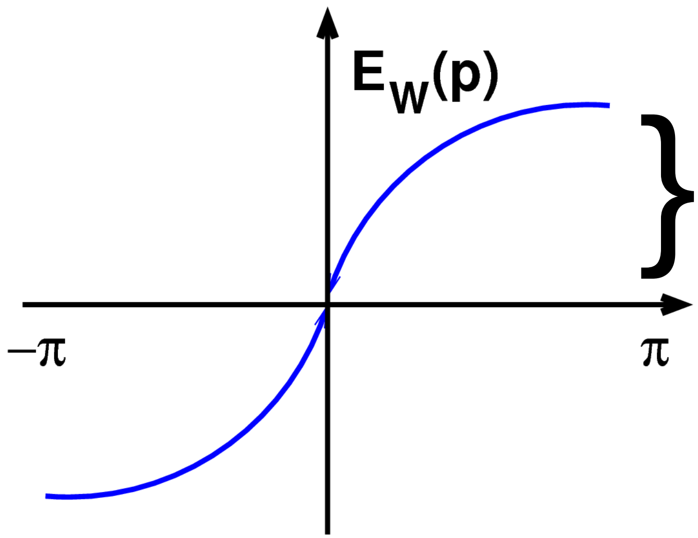
$$S_{Wilson} = \sum_{n,m} \bar{\psi}_n \Big[\frac{1}{2\kappa} \delta_{n,m} - \frac{1}{2} \sum_{\mu=0}^{3} \left((1+\gamma_\mu)\delta_{n,m-\mu} + (1-\gamma_\mu)\delta_{n,m+\mu} \right) \Big] \psi_m$$

$$\kappa = \frac{1}{8 + 2\hat{m}}$$

dispersion relation for Wilson fermions

$$M_W(p) = \left(\hat{m} + \sum_{\mu=0}^{3} (1 - \cos p_\mu) \right)^2 + \sum_{\mu=0}^{3} \sin^2 p_\mu$$

$= 2$ for $p_\mu = \pm\pi$



$= 2,\ 4,\ \ldots$

Wilson fermions remove doublers but break fundamental symmetries of continuum QCD (chiral symmetry): distortion of particle spectrum

Nielsen-Ninomiya theorem: any 4-d lattice discretization scheme for fermions either introduces doublers or breaks chiral symmetry

Phys. Lett B105 (1981) 219

staggered fermions $\Longleftrightarrow$ Kogut-Susskind fermions:

J.B. Kogut and L. Susskind, Phys. Rev. D11 (1975) 395

"naïve" fermion action:

$$S_{naive} = \sum_{n,m} \bar{\psi}_n \left[\frac{1}{2} \sum_{\mu=0}^{3} \gamma_\mu \left(\delta_{n,m-\mu} - \delta_{n,m+\mu} \right) + \hat{m}\delta_{n,m} \right] \psi_m$$

$$\psi_n \equiv \psi_n^{\alpha,i} \quad , \quad \alpha = 0, ..., 3 \quad , \quad i = 1, .., n_f$$

perform variable transformation on fermion fields:

$$\psi' = \Gamma\psi \quad \bar{\psi}' = \bar{\psi}\Gamma^\dagger \qquad \Gamma = \gamma_0^{n_0}\gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}$$

$$\Gamma^\dagger = \gamma_3^{n_3}\gamma_2^{n_2}\gamma_1^{n_1}\gamma_0^{n_0}$$

$\Longrightarrow$ mass term is invariant: $\bar{\psi}'\psi' = \bar{\psi}\psi$

kinetic term changes:

$$\bar{\psi}'_n\gamma_\mu\psi'_{n\pm\hat{\mu}} = \bar{\psi}_n\Gamma^\dagger\gamma_\mu\tilde{\Gamma}\psi_{n\pm\hat{\mu}} \qquad \tilde{\Gamma} = \gamma_0^{n_0}...\gamma_\mu^{n_\mu\pm1}..\gamma_3^{n_3}$$

all gamma matrices appear an even number of times in

$\Gamma^\dagger \gamma_\mu \tilde{\Gamma}$ i.e., this product is just +/- 1.

$$\Gamma^\dagger \gamma_\mu \tilde{\Gamma} = (-1)^{x_0 + \ldots x_{\mu-1}} \cdot 1_{spinor}$$

$$i = 1, ..n_f$$

$$\bar{\psi}^{\alpha,i} \gamma_{\alpha\beta} \psi^{\beta,i} \;\Rightarrow\; \eta_{n,\mu} \bar{\psi}^{\alpha,i} \psi^{\alpha,i} \qquad \alpha,\ \beta = 1, ..4$$

with $\eta_{n,\mu} = (-1)^{n_0 + \ldots n_{\mu-1}}$

the action now is diagonal in flavor AND spinor space!!

$$Z = \int \prod dU_{n,\mu} \left[\det\ M_{KS}\right]^{4n_f} \mathrm{e}^{-S_G}$$

J.B. Kogut and L. Susskind, Phys. Rev. D11 (1975) 395

drop 3 out of 4 "spinor" components; reduce fermion doubling from 16 to 4

$n_f = 1$ (drop 3 components):

$$Z = \int \prod dU_{n,\mu} \, [\det \; M_{KS}] \, e^{-S_G}$$

describes in the continuum limit a 4-flavor theory

$$M_{KS} = m \cdot 1 + D_{KS}$$

$$(M_{KS})^{nm} = \frac{1}{2} \sum_{\mu=0}^{3} \eta_{n,\mu} \left(U_{n,\mu} \delta_{n,m-\mu} - U^{\dagger}_{n-\mu,\mu} \delta_{n,m+\mu} \right)$$

$\eta_{n,\mu}$ does not change sign when 'moving' in μ direction

$\Rightarrow$ $$D_{KS} = \begin{pmatrix} 0 & D_{eo} \\ D_{oe} & 0 \end{pmatrix}$$

with $D_{oe} = -D^{\dagger}_{eo}$

i.e. $D^{\dagger} = -D$

eigenvalues are purely imaginary and
come in complex conjugate pairs $\Rightarrow$ $\det M_{KS} \geq 0$

Monte Carlo Simulations

Observables = Expectation values

$$Z = \int \mathcal{D}\mathcal{A}\mathcal{D}\psi\mathcal{D}\bar{\psi}\ \mathrm{e}^{-S_E} \qquad \langle \mathcal{O} \rangle = \frac{1}{Z}\int \mathcal{D}\mathcal{A}\mathcal{D}\psi\mathcal{D}\bar{\psi}\ \mathcal{O}\mathrm{e}^{-S_E}$$

integrate out fermions:

$$Z = \int \mathcal{D}\mathcal{A}\mathcal{D}\psi\mathcal{D}\bar{\psi}\ \mathrm{e}^{-S_E} = \int \mathcal{D}\mathcal{A} \prod_f \mathrm{det} M(m_f) \mathrm{e}^{-S_G} = \int \mathcal{D}\mathcal{A}\ \mathrm{e}^{-S}$$

$$S = S_G - \sum_f \mathrm{Tr}\ \ln M(m_f)$$

distribute gauge field configurations $\{\mathcal{A}^{(i)}\}_{i=1}^{N_{tot}}$ according to the probability distribution $P(\{\mathcal{A}\}) = Z^{-1}\mathrm{e}^{-S(\{\mathcal{A}\})}$

Monte Carlo algorithms (detailed balance) generates Markov chain

calculate expectation values: $\langle \mathcal{O} \rangle = \lim_{N_{tot}\to\infty} \frac{1}{N_{tot}} \sum_{i=1}^{N_{tot}} \mathcal{O}(\{\mathcal{A}^{(i)}\})$

Dealing with the fermion determinant

partition function again:

$$Z(V,T) = \int \mathcal{D}\mathcal{A} \int \prod_n \mathrm{d}\psi_n \mathrm{d}\bar{\psi}_n \ \mathrm{e}^{\bar{\psi}_n M(\mathcal{A},m_q)_{nm}\psi_m} \ \mathrm{e}^{-S_G}$$

fermions (anti-commuting)

"fermion matrix"

$$= \int \mathcal{D}\mathcal{A} \ \det M(\mathcal{A}, m_q) \ \mathrm{e}^{-S_G}$$

"importance sampling"

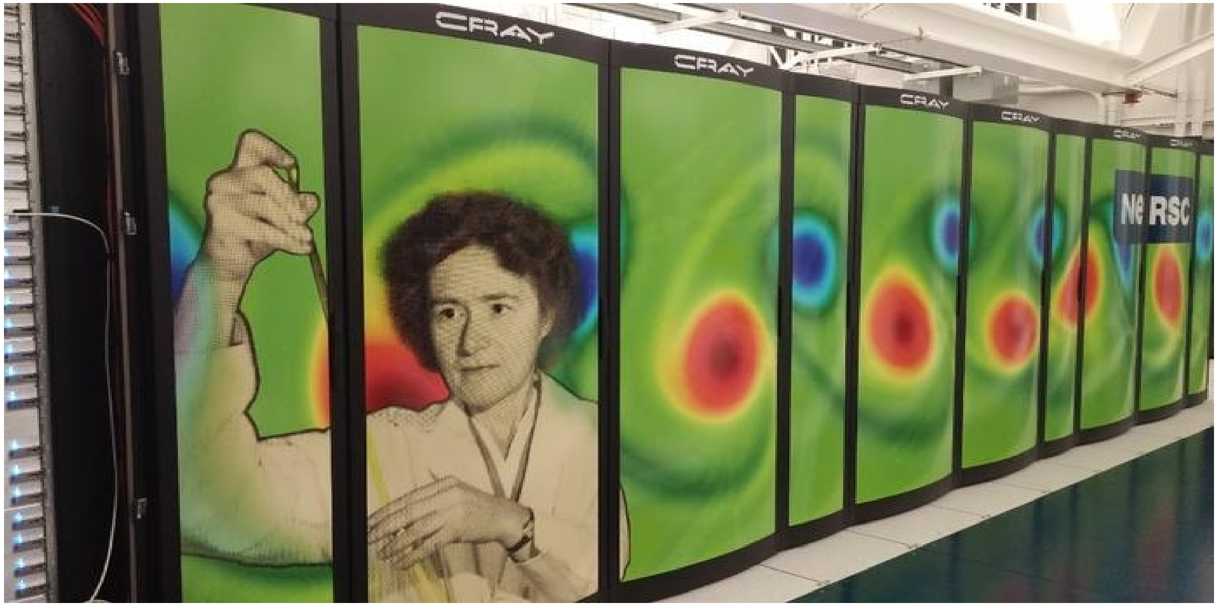
$$\det M(\mathcal{A}, m_q) = \int \prod_n \mathrm{d}\phi_n \ \mathrm{e}^{-\sum_{nm} \phi_n^* M^{-1}_{nm}(\mathcal{A},m_q)\phi_m}$$

bosons (commuting)

– need $x_n = M^{-1}_{nm}\phi_m$

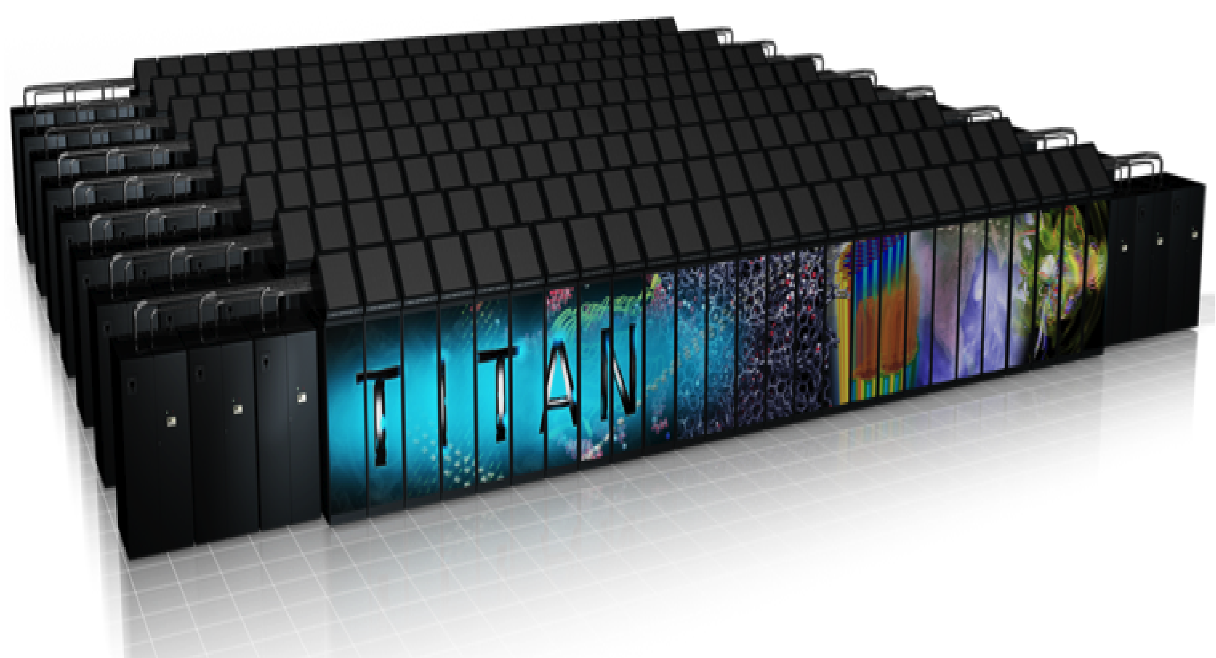
– solve $M_{nm}x_m = \phi_n$

Computational resources for Lattice QCD



Cori@NERSC
Intel Xeon Phi processor
Knights Landing

9300 compute nodes
#5, top500, 2016
peak: 29 Petaflops
$=29 \times 10^{15}$ Flops

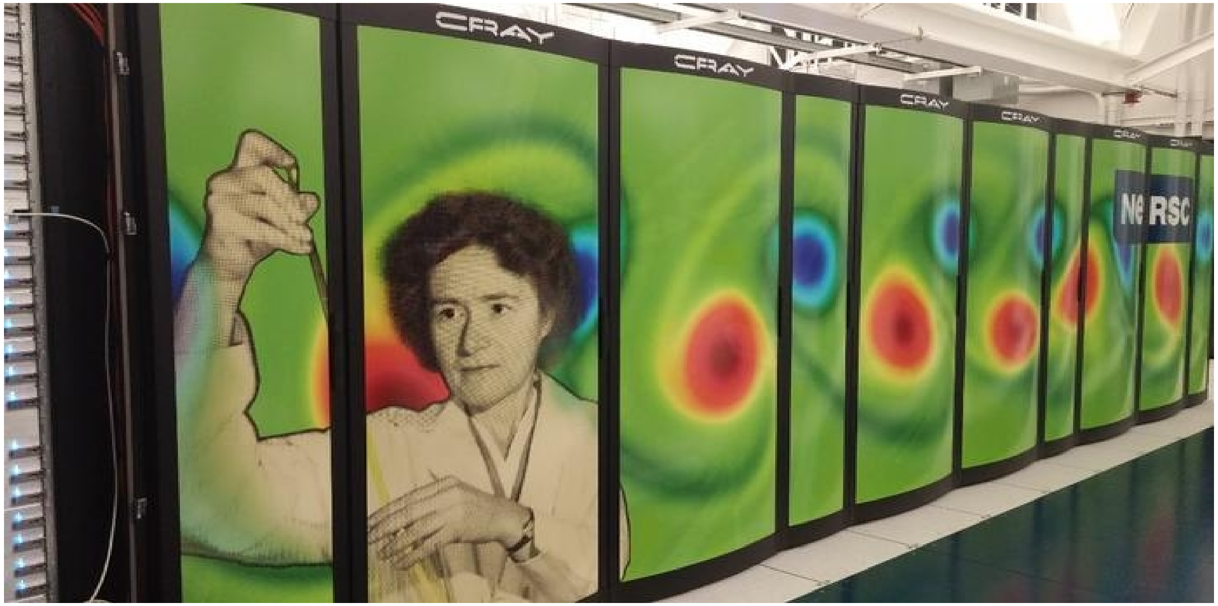


Titan@ORNL
Cray with NVIDIA
graphics cards K20X
18688 GPUs

#3, top500, 2016
peak: 20 Petaflops
=20 Mill. Gflops

QCD as a video game,
G.I. Egri et al, Comp. Phys. Com, **177**, 631 (2007)

Computational resources for Lattice QCD



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2016: speed increased by $\mathbf{10^{10}}$, i.e. a factor 2.5 every year

1980/81:
first lattice calculation of an equation of state for gluon matter in Bielefeld

Thermodynamics of strong-interaction matter

reminder:

Euclidean path integrals and thermodynamics (in quantum mechanics)

Time evolution operator: $$U(t,t') = \exp\left(-\frac{i}{\hbar}(t-t')H\right)$$

$$\begin{aligned}\psi_n(x,t) &\equiv \langle x|U(t,0)|\psi_n\rangle \\ &= \mathrm{e}^{-iE_nt/\hbar}\langle x|\psi_n\rangle\end{aligned}$$

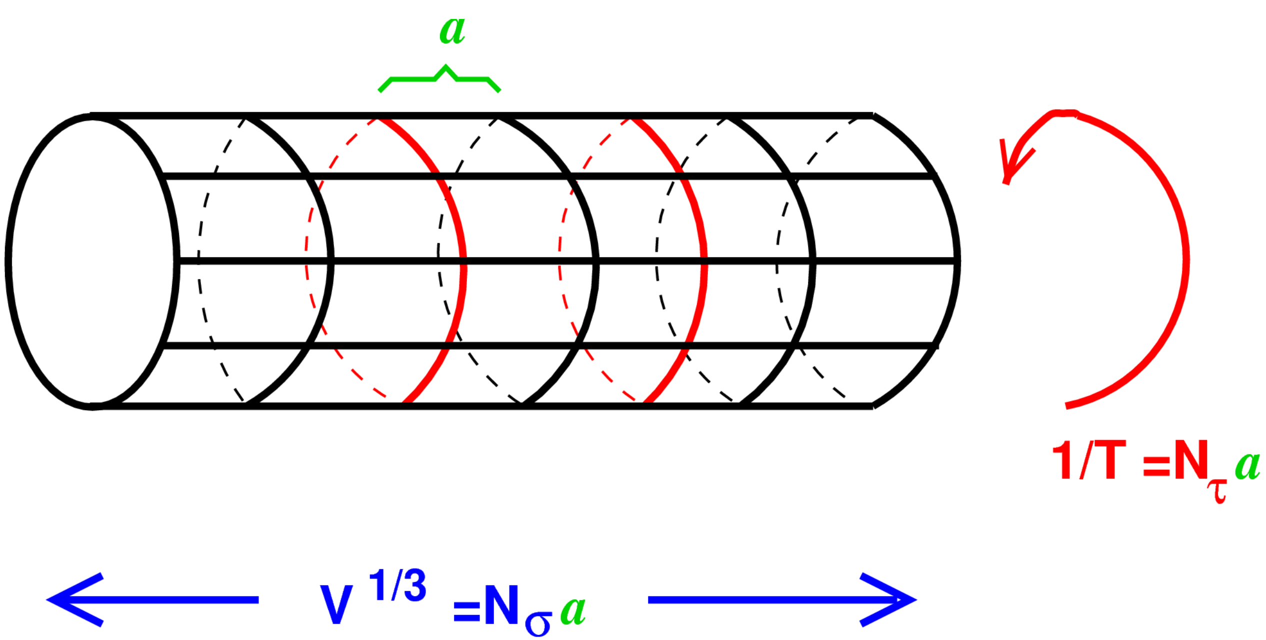
Green's functions: $$\begin{aligned}G(y,t'';x,t') &\equiv \langle y|U(t'',t')|x\rangle \\ &= \sum_n \mathrm{e}^{-\frac{i}{\hbar}E_n(t''-t)}\psi_n(y)\psi_n^*(x)\end{aligned}$$

Special case – periodic paths: y = q(t) = q(0) = x

$$\begin{aligned}\tilde{Z}(t) &= \int dx\; G(x,t;x,0) = \int dx \langle x|\mathrm{e}^{-itH/\hbar}|x\rangle \\ &= \int dx \sum_n \mathrm{e}^{-\frac{i}{\hbar}E_nt}\psi_n(x)\psi_n^*(x) = \sum_n \mathrm{e}^{-\frac{i}{\hbar}E_nt} = \mathrm{Tr}\;\mathrm{e}^{-itH/\hbar}\end{aligned}$$

$$\tilde{Z}(it) = Z(\beta)\quad \text{with}\quad t=-i\hbar/T\;\;,\;\; \beta = 1/T$$

Simulating strongly interacting matter on a discrete space-time grid (lattice QCD)



the lattice: $N_\sigma^3 \times N_\tau$

lattice spacing: a

temperature: $T = 1/N_\tau a$

partition function:

$$Z(V,T,\mu) = \int \mathcal{D}\mathcal{A}\mathcal{D}\psi\mathcal{D}\bar{\psi}\; \mathrm{e}^{-S_E}$$

<u>bulk thermodynamics:</u>

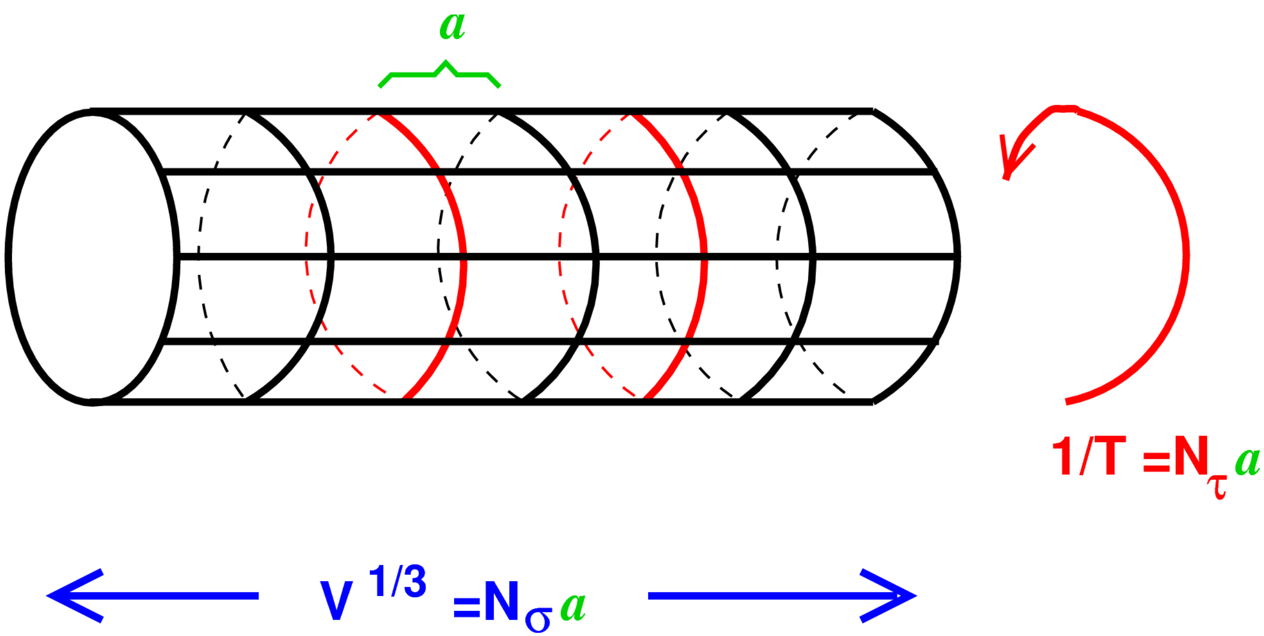
$$\frac{p}{T^4} = \frac{1}{VT^3}\ln Z$$

$$\frac{\epsilon}{T^4} = -\frac{1}{VT^4}\frac{\partial \ln Z}{\partial T^{-1}}$$

$$\frac{n_B}{T^3} = \frac{1}{VT^3}\frac{\partial \ln Z}{\partial \mu_B/T}$$

$$\frac{\chi_B}{T^2} = \frac{1}{VT^3}\frac{\partial^2 \ln Z}{\partial (\mu_B/T)^2}$$

Simulating strongly interacting matter on a discrete space-time grid (lattice QCD)



the lattice: $N_\sigma^3 \times N_\tau$

lattice spacing: a

temperature: $T = 1/N_\tau a$

<u>phase structure:</u>

order parameter: $\frac{\langle \bar\psi \psi \rangle_l}{T^3} = \frac{1}{VT^3} \frac{1}{4} \frac{\partial \ln Z}{\partial m_l/T}$

chiral susceptibility: $\frac{\chi_l}{T^2} = \frac{\partial \langle \bar\psi \psi \rangle_l / T^3}{\partial m_l/T}$

<u>bulk thermodynamics:</u>

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Calculating the equation of state on lines of constant physics (LCPs)

pressure (not an expectation value): $\frac{p}{T^4} = \frac{1}{VT^3} \ln Z$

trace anomaly (sometimes called interaction measure):

$$\frac{\epsilon - 3p}{T^4} = T\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{p}{T^4}\right) = \left(a\frac{\mathrm{d}\beta}{\mathrm{d}a}\right)_{LCP} \frac{\partial p/T^4}{\partial \beta}$$

$$= \left(\frac{\epsilon - 3p}{T^4}\right)_{gluon} + \left(\frac{\epsilon - 3p}{T^4}\right)_{fermion} + \left(\frac{\epsilon - 3p}{T^4}\right)_{\hat{m}_s/\hat{m}_l}$$

pressure (reconstructed): $\left.\frac{p}{T^4}\right|_{\beta_0}^{\beta} = \int_{T_0}^{T} \mathrm{d}T\, \frac{1}{T}\left(\frac{\epsilon - 3p}{T^4}\right)$

lines of constant physics (LCP):

need T-scale $aT = 1/N_\tau$ and its relation to the gauge coupling $a \equiv a(\beta)$

LCP: for given β choose $m_{u/d}$, m_s such that the "known hadron spectrum gets reproduced at T=0"

Calculating the equation of state on lines of constant physics (LCPs)

trace anomaly: $$\frac{\epsilon - 3p}{T^4} = T\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{p}{T^4}\right) \equiv a\left(\frac{N_\tau}{N_\sigma}\right)^3 \frac{\mathrm{d}\ln Z(T,V)}{\mathrm{d}a}$$ $$T \equiv \frac{1}{N_\tau a}$$

$$\frac{\epsilon - 3p}{T^4} \equiv \frac{\Theta_G^{\mu\mu}(T)}{T^4} + \frac{\Theta_F^{\mu\mu}(T)}{T^4}$$

$\langle \cdots \rangle_{0(\tau)}$
expectation values at zero (finite) temperature

$$\frac{\Theta_G^{\mu\mu}(T)}{T^4} = R_\beta \left[\langle s_G \rangle_0 - \langle s_G \rangle_\tau\right] N_\tau^4$$

$$\frac{\Theta_F^{\mu\mu}(T)}{T^4} = -R_\beta R_m [2m_l \left(\langle \bar\psi\psi \rangle_{l,0} - \langle \bar\psi\psi \rangle_{l,\tau}\right)$$
$$+ m_s \left(\langle \bar\psi\psi \rangle_{s,0} - \langle \bar\psi\psi \rangle_{s,\tau}\right)] N_\tau^4$$

beta-functions:

$$R_\beta(\beta) = \frac{r_1}{a}\left(\frac{\mathrm{d}(r_1/a)}{\mathrm{d}\beta}\right)^{-1}$$

$$R_m(\beta) = \frac{1}{m_s(\beta)}\frac{\mathrm{d}m_s(\beta)}{\mathrm{d}\beta}$$

scale setting: r_1 -scale: $$r^2 \frac{dV_{\bar q q}}{dr}\bigg|_{r_1} = 1$$

$$r_1 = 0.3106 \text{ fm}$$

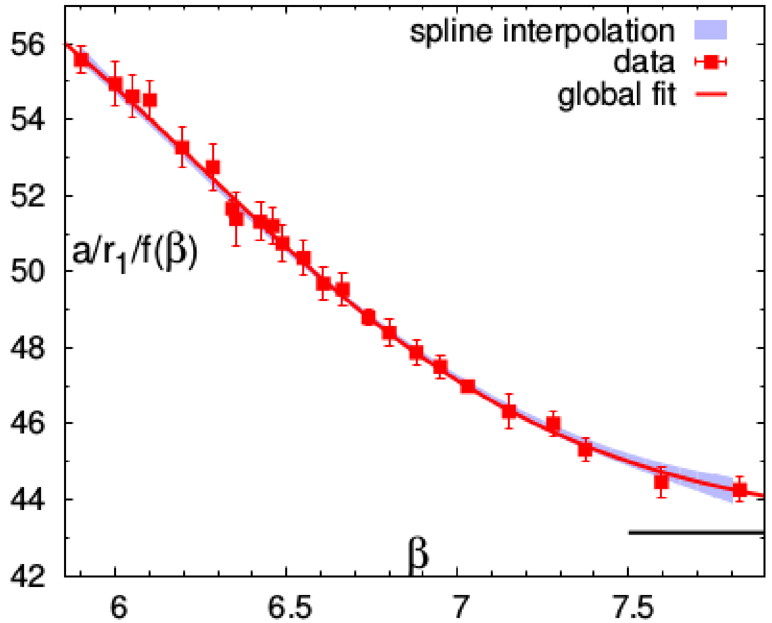
bare strange quark mass: $m_s(\beta)$

keep a strange hadron mass const.

$$m_{sH} r_1 \equiv m_{sH} a \cdot \frac{r_1}{a} = \text{const.}$$

A) – calculate the heavy quark potential $V_{\bar{q}q}(r/a)a$

– extract r_1/a

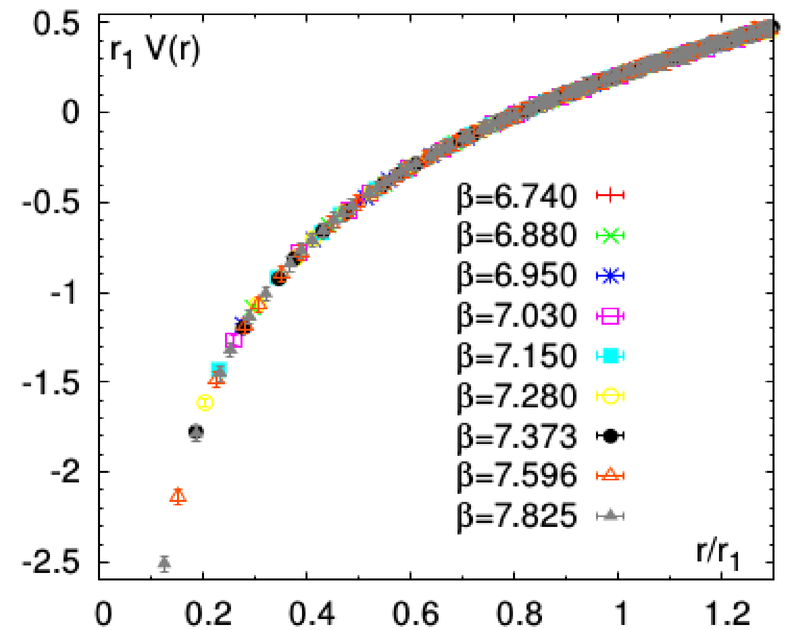


$$\frac{a}{r_1} = \frac{c_0 f(\beta) + c_2(10/\beta) f^3(\beta)}{1 + d_2(10/\beta) f^2(\beta)}$$

$$f(\beta) = \left(\frac{10 b_0}{\beta}\right)^{-b_1/(2b_0^2)} \exp(-\beta/(20 b_0))$$

$$b_0 = \frac{9}{16\pi^2} \quad , \quad b_1 = \frac{1}{4\pi^4}$$

Heavy Quark potential



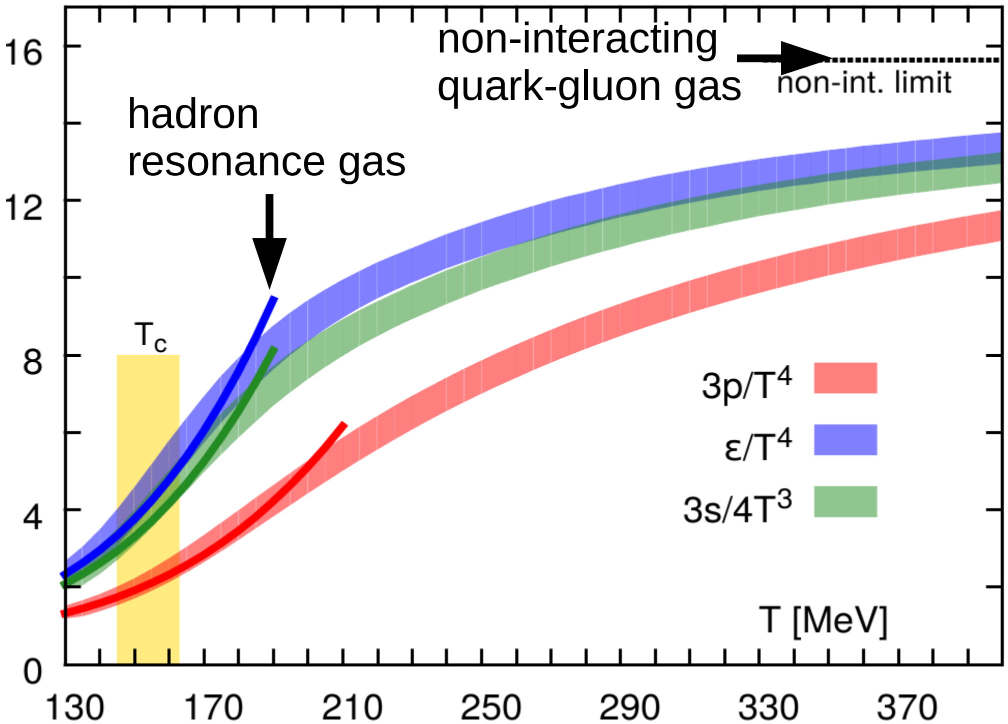
B) – calculate a hadron mass $m_H a$

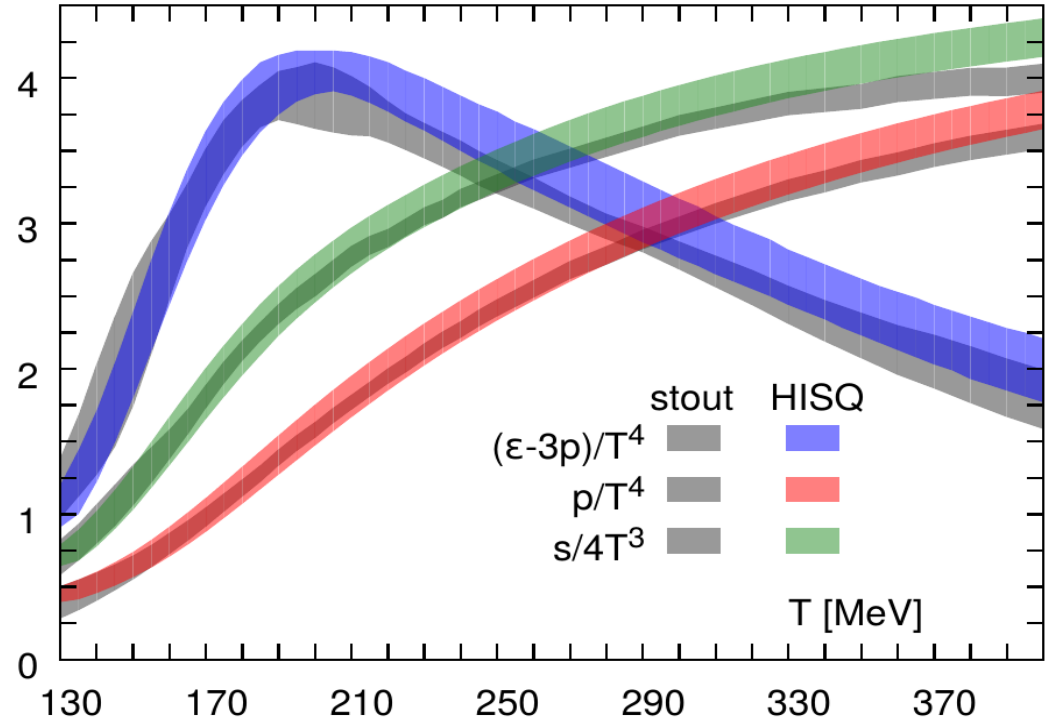
– tune the bare quark mass(es) such that $m_H r_1 = m_H a \cdot r_1/a$ takes on its physical value

$\rightarrow$ $m_l(\beta),\ m_s(\beta)$

Equation of state of (2+1)-flavor QCD

pressure, **entropy** & **energy** density





A. Bazavov et al. (hotQCD) ,
Phys. Rev. D90 (2014) 094503

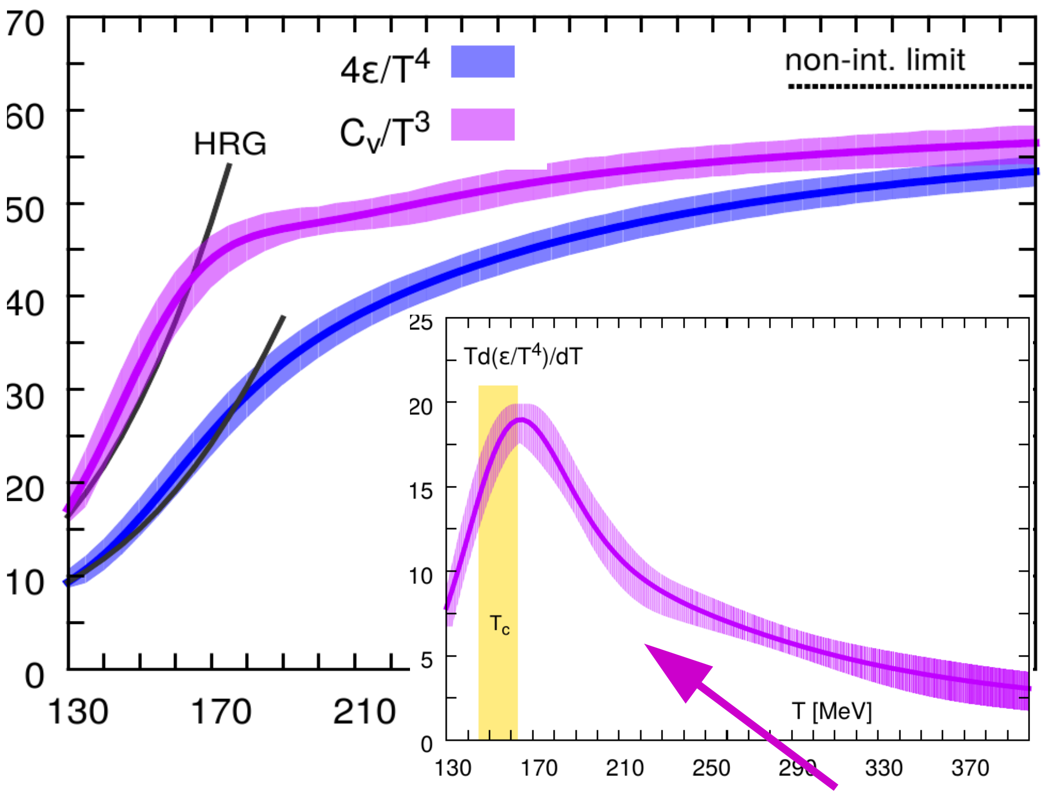
– improves over earlier hotQCD calculations:
A. Bazavov et al., Phys. Rev. D80, 014504 (2009)

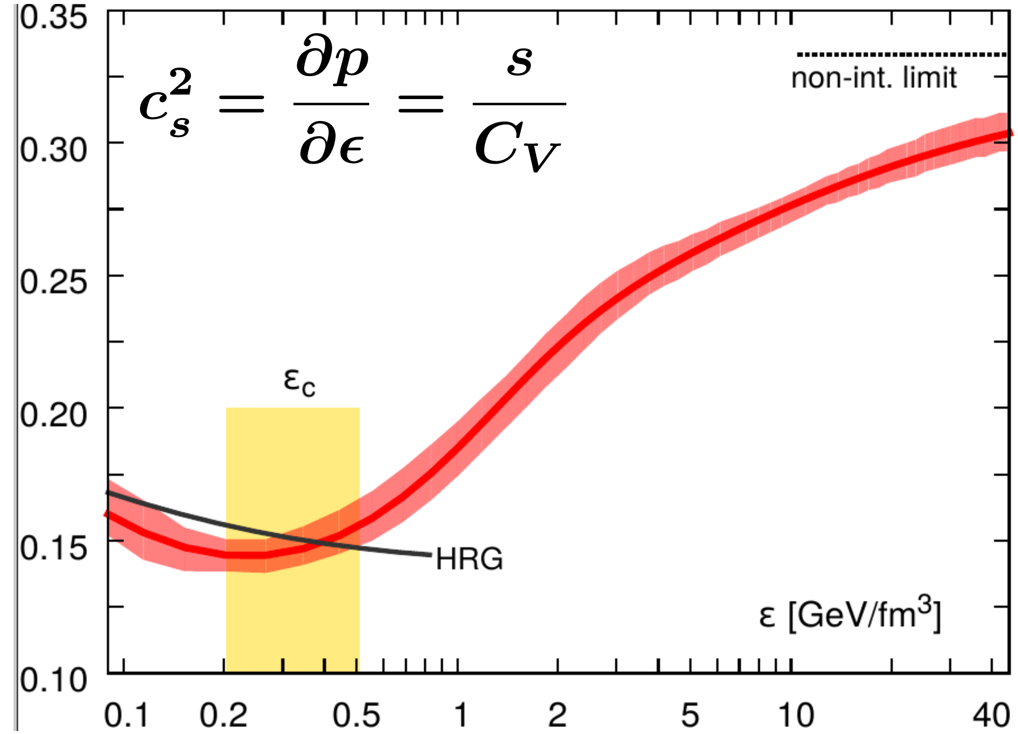
– consistent with results from Budapest-Wuppertal (stout): S. Borsanyi et al., PL B730, 99 (2014)

– up to the crossover region the QCD EoS agrees quite well with hadron resonance gas (HRG) model calculations; **However**, QCD results are systematically above HRG

Equation of state of (2+1)-flavor QCD

specific heat & speed of sound





$$C_V = \left.\frac{\partial \epsilon}{\partial T}\right|_V \equiv \left(4\frac{\epsilon}{T^4} + T \left.\frac{\partial(\epsilon/T^4)}{\partial T}\right|_V \right) T^3$$

softest point:

$$\left(c_s^2\right)_{min} \simeq \frac{1}{2} \left(c_s^2\right)^{free\ gas}$$

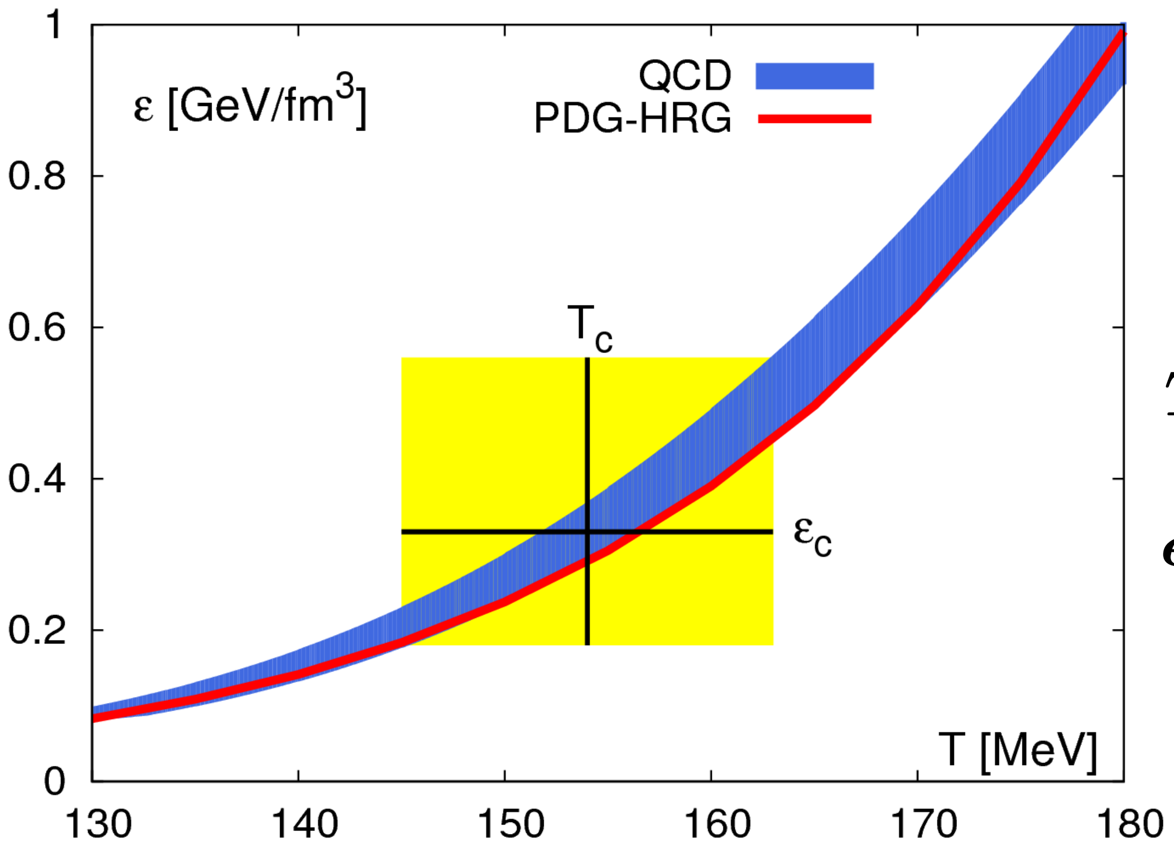
large regular back-ground at high-T

dominant singular part of the specific heat

A. Bazavov et al. (hotQCD) , Phys. Rev. D90 (2014) 094503

Crossover transition parameters

PDG: Particle Data Group hadron spectrum



$$T_c = (154 \pm 9)\ \mathrm{MeV}$$

$$\epsilon_c = (0.34 \pm 0.16)\mathrm{GeV/fm^3}$$

compare with:

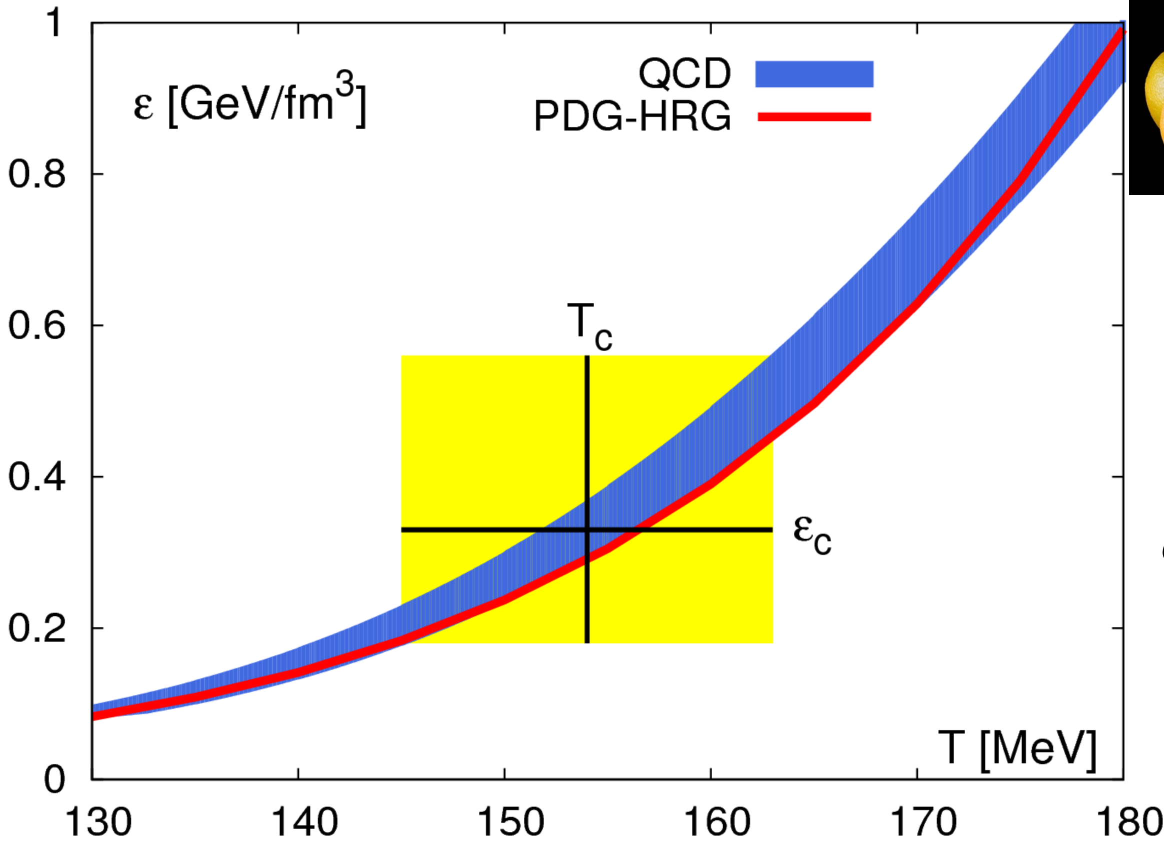
$$\epsilon^{\mathrm{nucl.\ mat.}} \simeq 150\ \mathrm{MeV/fm^3}$$

$$\epsilon^{\mathrm{nucleon}} \simeq 450\ \mathrm{MeV/fm^3}$$

A. Bazavov et al. (hotQCD) ,
Phys. Rev. D90 (2014) 094503

Crossover transition parameters

PDG: Particle Data Group hadron spectrum



dense packing of spheres (DPS)

$$\epsilon^{\mathrm{DPS}} = 0.74\epsilon^{\mathrm{nucleon}} \simeq 0.33\ \mathrm{GeV/fm^3}$$

$$(R_p \simeq 0.8\mathrm{fm})$$

$$T_c = (154 \pm 9)\ \mathrm{MeV}$$

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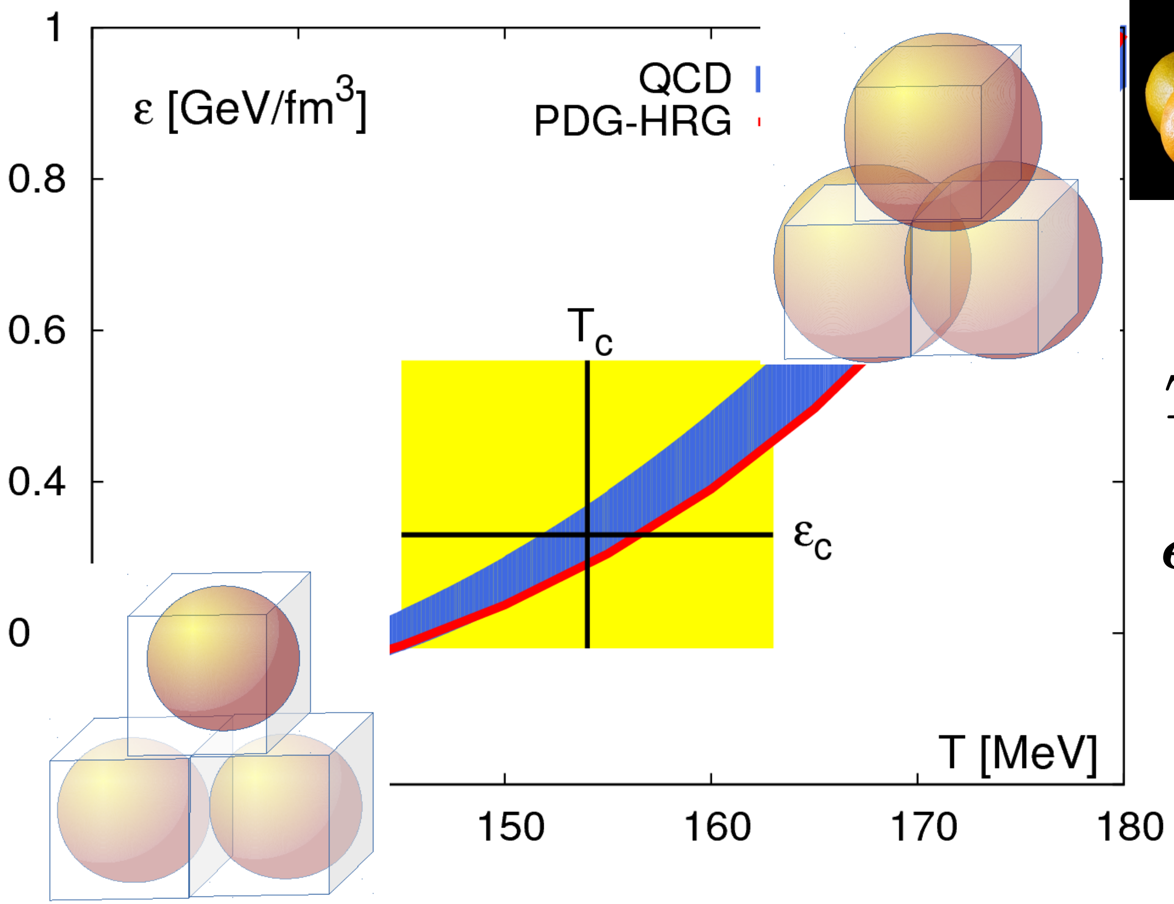

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overlapping hadrons = QGP ??

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