Lattice QCD at non-zero temperature and density

Frithjof Karsch
Bielefeld University & Brookhaven National Laboratory

- QCD in a nutshell, non-perturbative physics, lattice-regularized QCD, Monte Carlo simulations
- the phase diagram on strongly interacting matter, chiral symmetry restoration, the equation of state
- finite density QCD, cumulants of conserved charge fluctuations, thermal masses & transport properties

Literature

- M. Creutz, Quarks, Gluons and Lattices
- H. Rothe, Lattice Gauge Theories
- T. DeGrand, C. DeTar, Lattice Methods for Quantumchromodynamics
- C. Gattringer and C.B. Lang, Quantum Chromodynamics on the Lattice
- I. Montvay, G. Muenster, Quantum Fields on a Lattice
- J. Zinn-Justin, Quantum Field Theory and Critical Phenomena
- H.T. Ding, FK, S. Mukherjee, Thermodynamics of strong-interaction Matter from Lattice QCD, Int. J. Mod. Phys. E24 (2015) 153007

Motivation

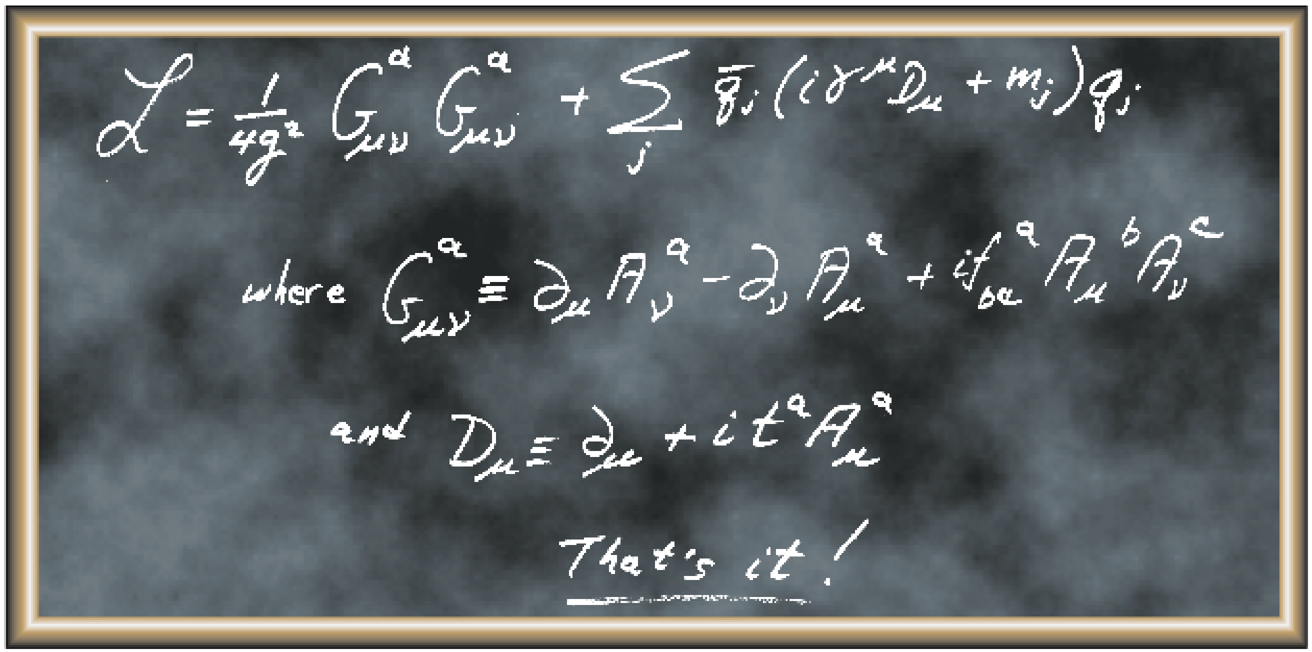
Why lattice (gauge) theory?

Many phenomena in physics are of 'non-perturbative' nature

- Spontaneous symmetry breaking:
  - Phase transitions (spontaneous magnetization, Ising model)
  - Higgs mechanism
  - chiral symmetry breaking
- The origin of mass, hadron spectra
  - $m_{u,d} \simeq 5$ MeV, but $m_p = 938$ MeV
- Long distance properties of Quantum Field Theories
  - confinement, screening

The theory of strong-interaction

Quantum Chromo Dynamics (QCD)



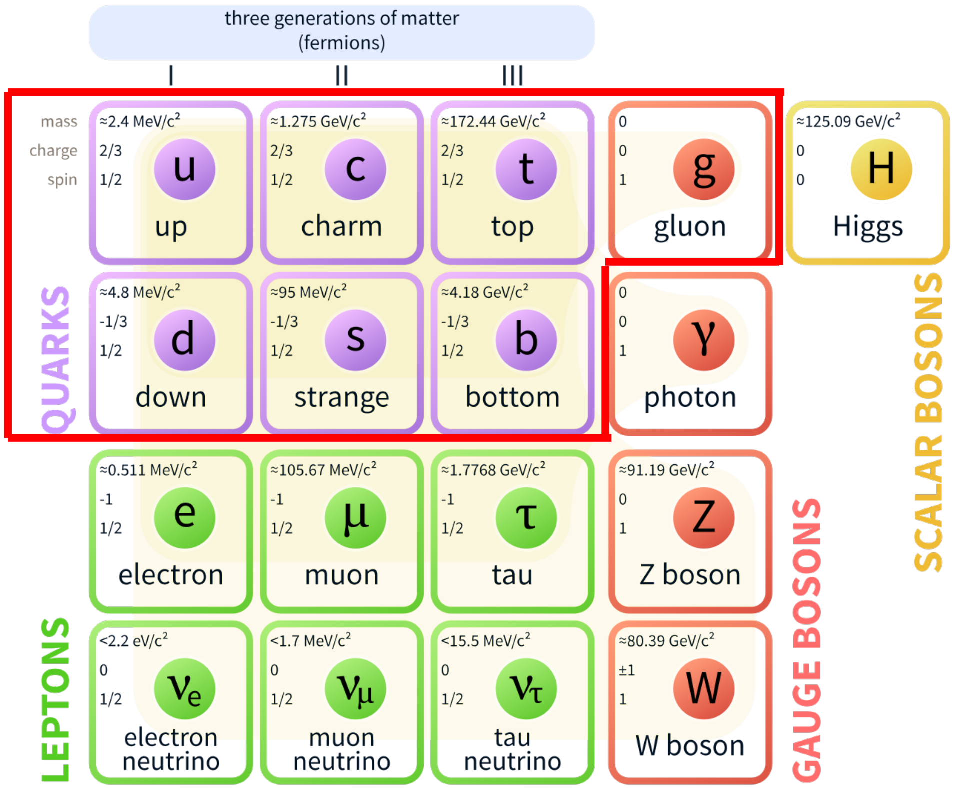
F. Wilczek, QCD made simple,
Phys. Today 53N8 (2000) 22

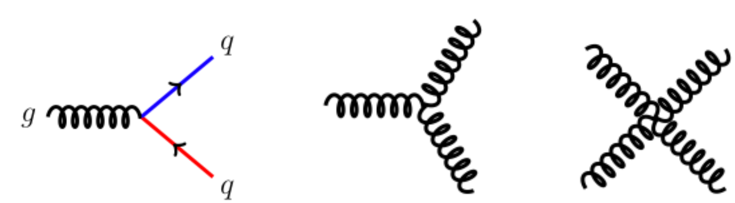
The theory of strong-interaction

Quantum Chromo Dynamics (QCD)

- non-abelian gauge theory, constructed in analogy to QED
- matter fields (quarks [fermions]) and force carrier (gluons [bosons])
- a new quantum number – color
  $N_c = r, b, g$
- quarks live in the fundamental representation of the SU(3) color group:
  - each quark species/flavor carries one of three possible colors
- gluons live in the adjoint representation; there are 8 different gluons
  - unlike the photon in QED the gluons not only interact with quarks but also among themselves
    (three- and four-gluon vertices)

Standard Model of Elementary Particles





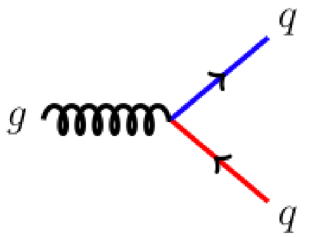
quark-gluon vertex

3-gluon and 4-gluon vertices vertex

The theory of strong-interaction

Quantum Chromo Dynamics (QCD)

– asymptotic freedom --



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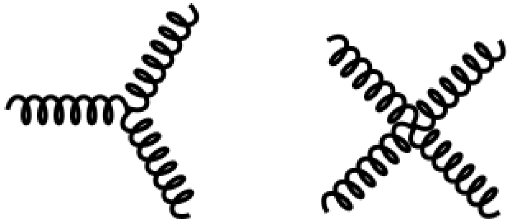
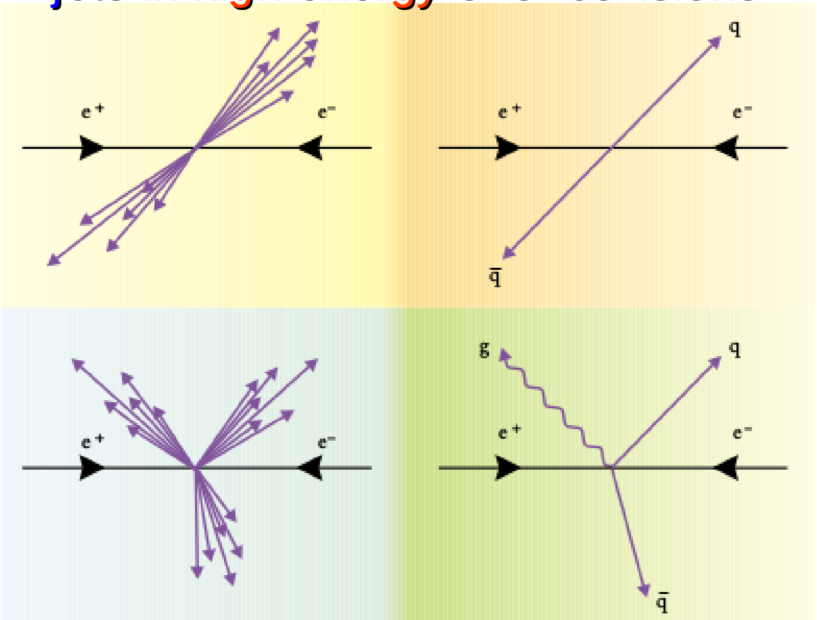


3-gluon and 4-gluon vertices vertex

jets in high energy e+e- collisions



F. Wilczek, QCD made simple,
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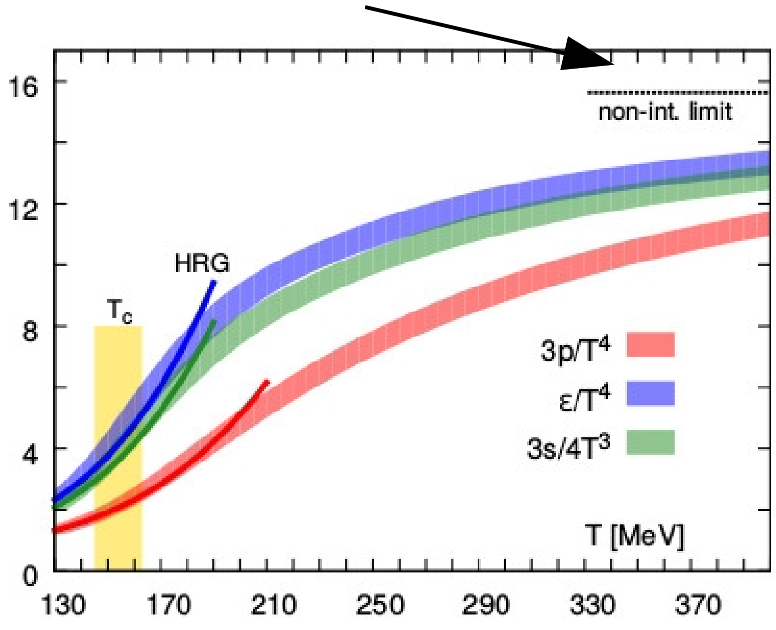
weakly interacting quarks& gluons at high-T

$$\epsilon/T^4 = (\epsilon/T^4)_{free} + c_2(T)g^2(T) + \ldots$$

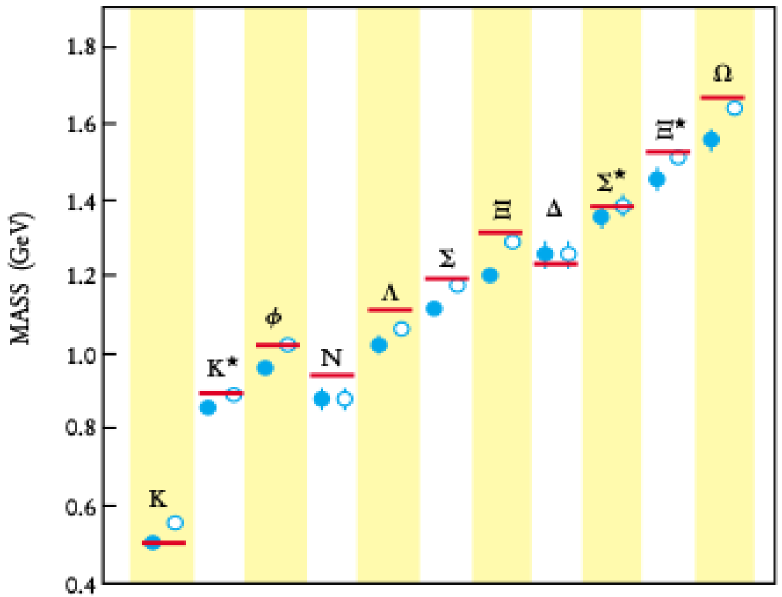


The theory of strong-interaction

Quantum Chromo Dynamics (QCD)

– confinement --

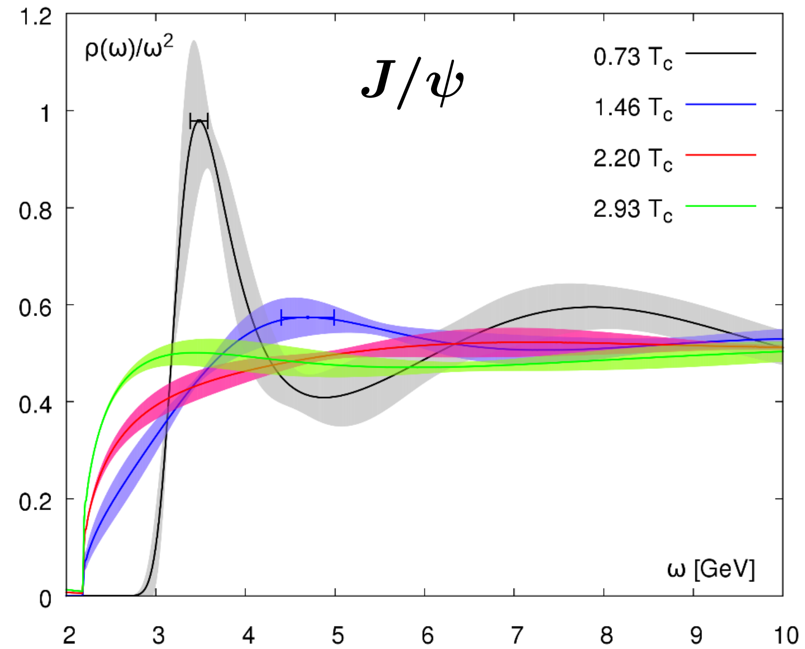
Hadron spectrum in the vacuum



F. Wilczek, QCD made simple, Phys. Today 53N8 (2000) 22

Y. Aoki et al, Nature 443 (2006) 675

Hadrons in a thermal heat bath

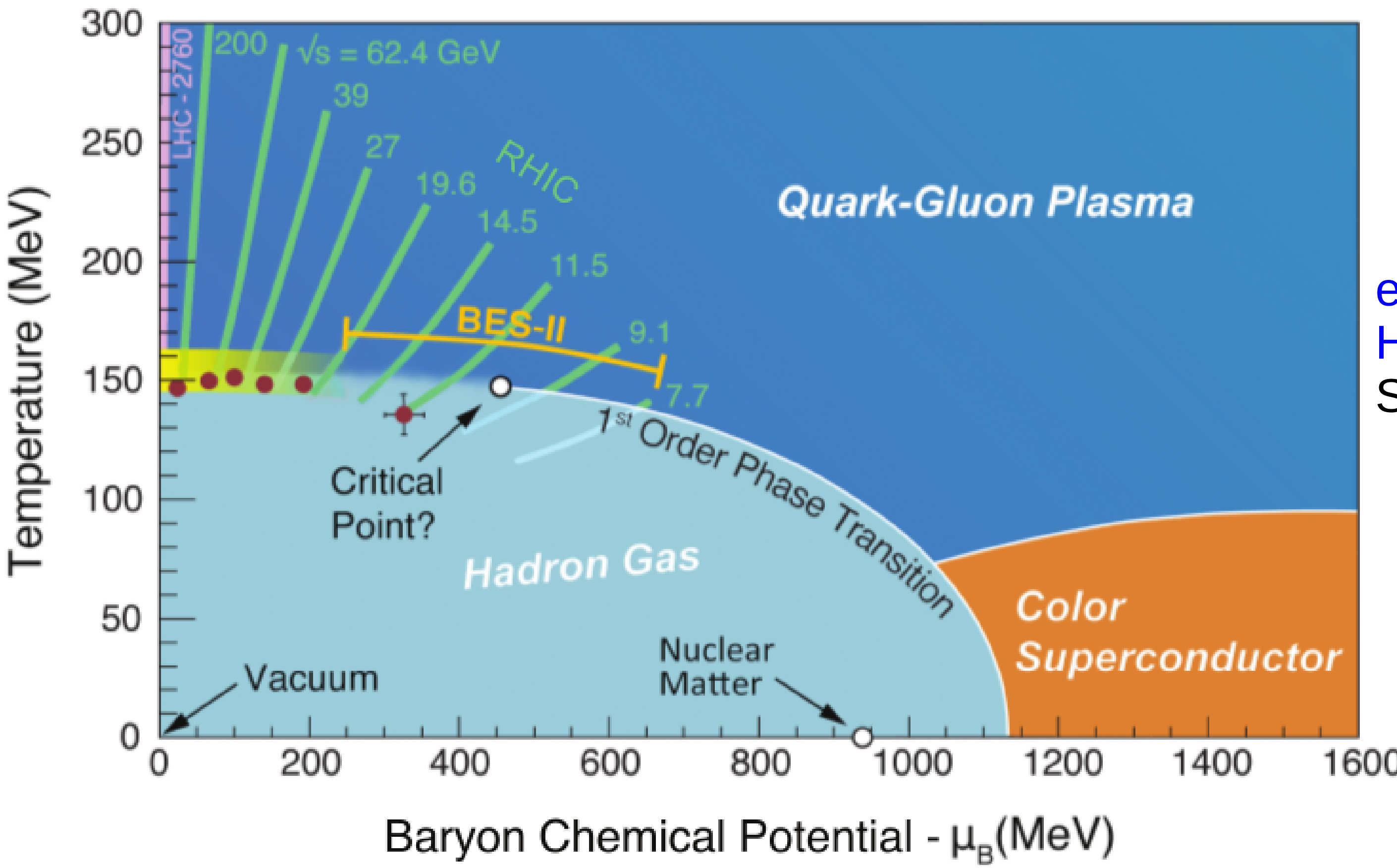


melting of heavy quark bound states

T. Matsui and H. Satz, Phys. Lett. B178 (1986) 416

H.-T. Ding et al., Phys. Rev. D86 (2012) 014509

Phases of strong-interaction matter



physics of the early universe

hot $T \sim 10^{12}\mathrm{K}$

experimentally accessible in Heavy Ion Collisions at SPC, RHIC, LHC, FAIR, NICA

may exist in the interior of compact stars,

dense $n_B \sim 10 n_{NM}$

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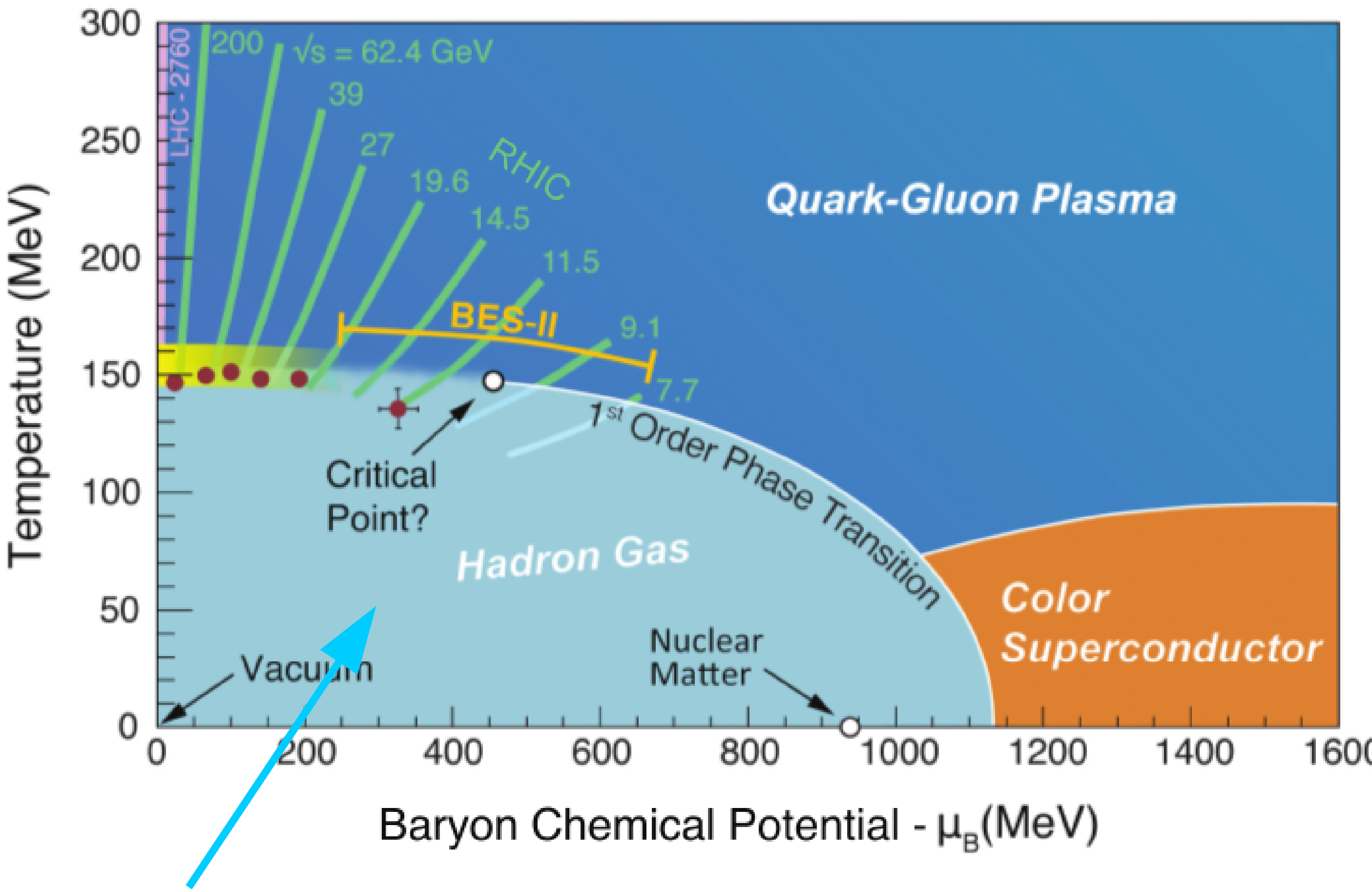

thermodynamics of a hadron resonance gas

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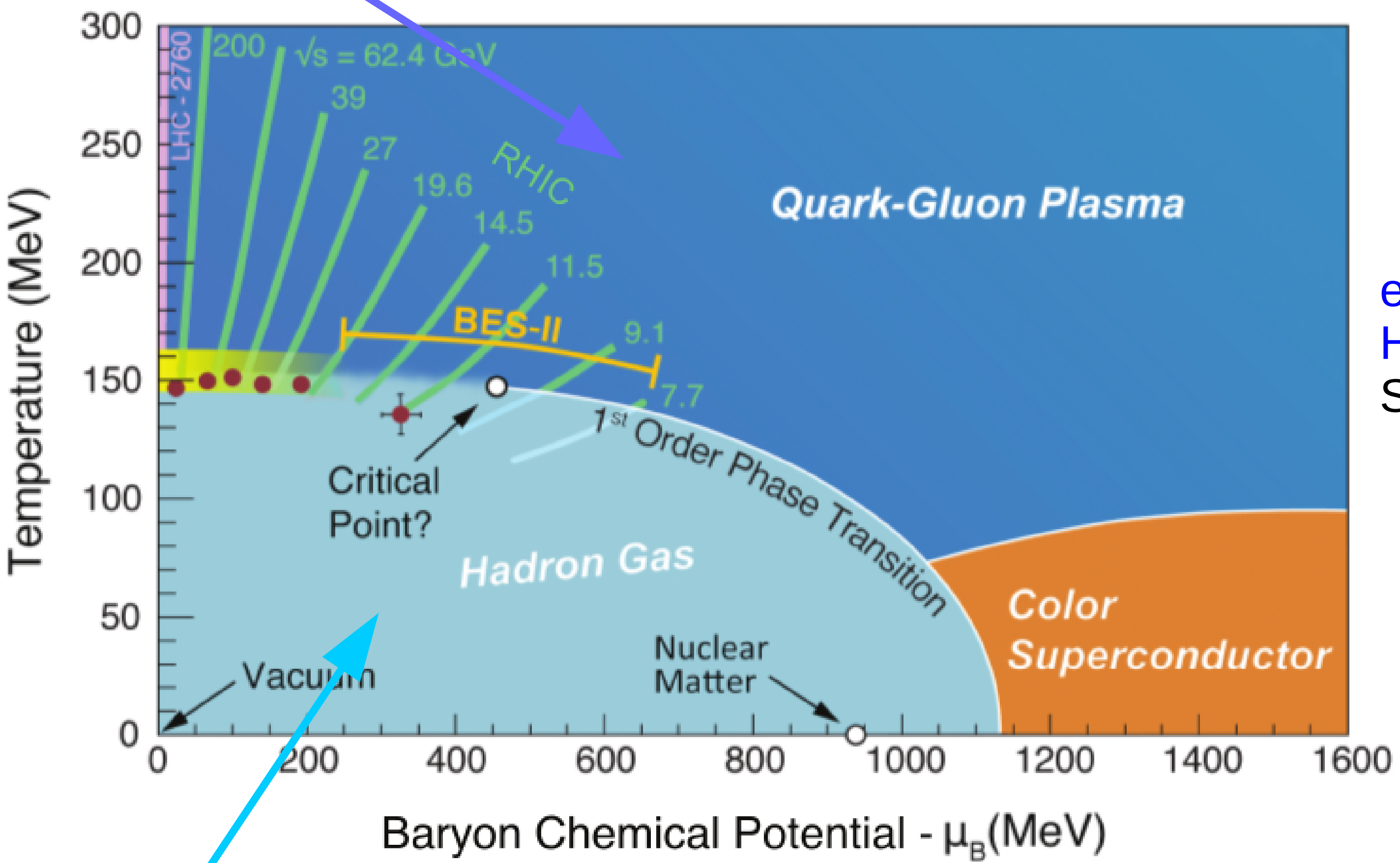
Phases of strong-interaction matter

thermodynamics of quarks and gluons:
high-T – QCD perturbation theory



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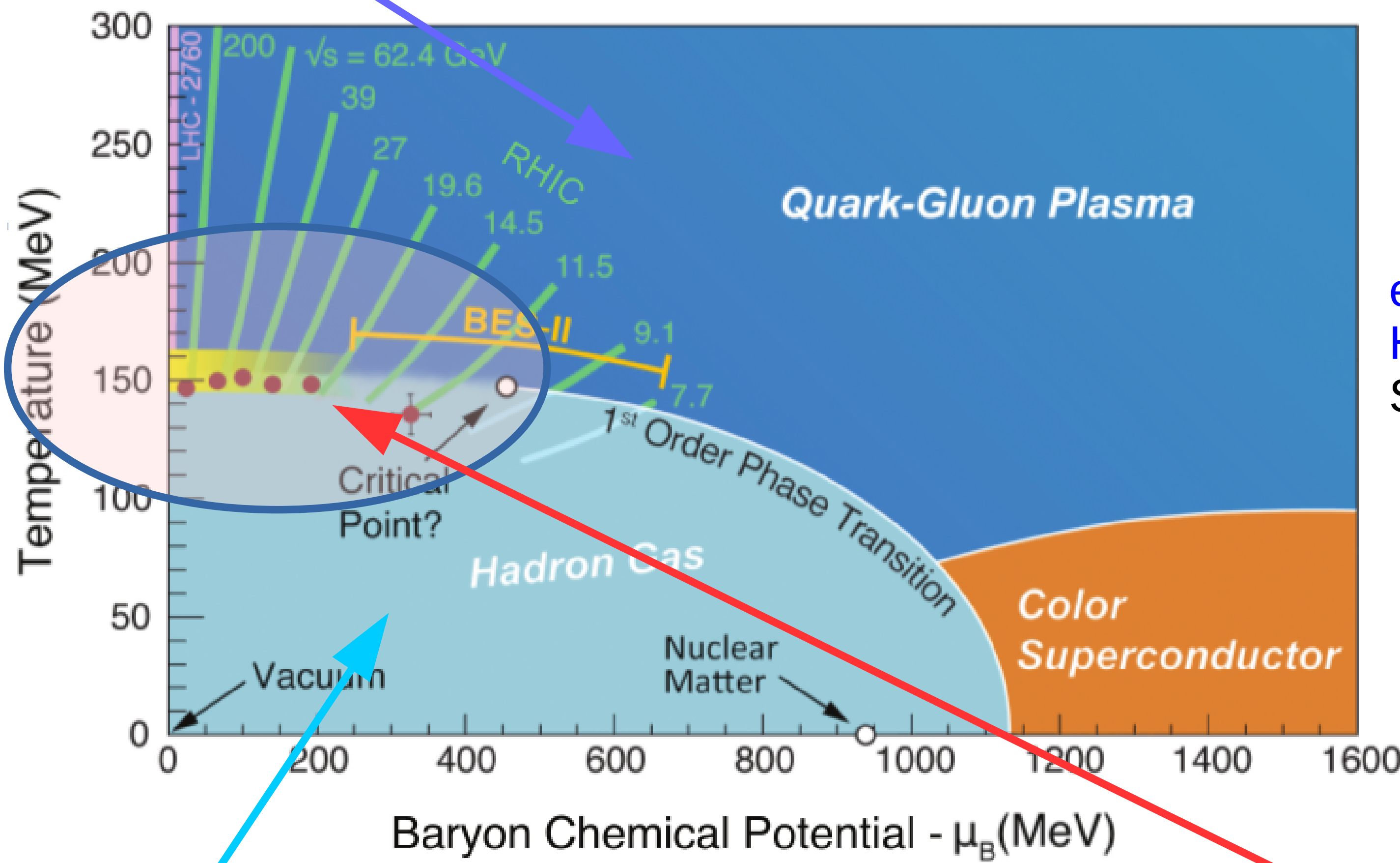

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may exist in the interior of compact stars,
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thermodynamics of a hadron resonance gas

strongly interacting, highly non-perturbative region close to phase boundaries
need for lattice QCD calculations

Quantum Chromo Dynamics (QCD)

– QCD is expected to explain ALL aspects of strong interaction physics

– a rich spectrum of hadrons (including the values of their masses)

– the peculiar role of the lightest hadrons → pions
– the mass splitting among parity partners
– the mass-splitting of pion and delta (a_0)

spontaneous breaking of chiral symmetries

– the absence of its basic constituents (quarks and gluons) from the experimentally observed particle spectrum

confinement

– the simple properties of deep inelastic scattering

asymptotic freedom

– the existence (?) of phase transitions

deconfinement & chiral symmetry resoration

Lattice regularized Quantum Chromo Dynamics (QCD)

– start with the Euclidean version of the QCD Lagrangian:

$$Z(g^2,\{m_f\}_{f=1}^{n_f}) \;=\; \int \mathcal{D}\mathcal{A}\mathcal{D}\psi\mathcal{D}\bar{\psi}\; \mathrm{e}^{-S_E} \;,\; S_E \equiv \int d^4x\; \mathcal{L}_E(x)$$

$$\gamma^k \to i\gamma_k^E,\; k-1,2,3$$

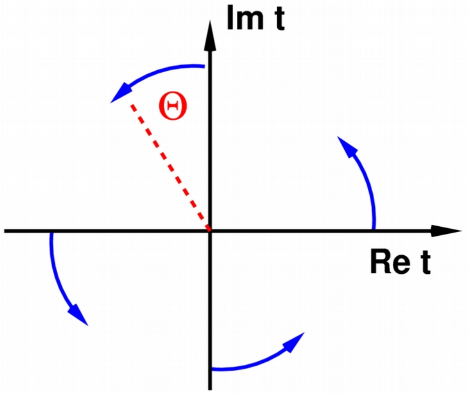
$$\gamma_0 \to \gamma_0^E$$

(suppress *E* again)

$$t \;=\; |t|\mathrm{e}^{-i\theta}$$

$$dt \;\to\; d|t|\mathrm{e}^{-i\theta}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \;\to\; \frac{\mathrm{d}}{\mathrm{d}|t|}\mathrm{e}^{i\theta}$$



$$\mathcal{L}_E = \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \bar{\psi}_{j,a}\left(\sum_{\nu=0}^{3}\gamma_\nu\left(\partial_\nu - i\frac{g}{2}\mathcal{A}^a_\nu\lambda^a\right) + m_j\right)^{a,b}\psi_{j,b}$$

field strength tensor: $F^a_{\mu\nu} = \partial_\mu\mathcal{A}^a_\nu - \partial^a_\nu\mathcal{A}^a_\mu - gf_{abc}\mathcal{A}^b_\mu\mathcal{A}^c_\nu$

gluons: real 4-vector $\mathcal{A}^a = (\mathcal{A}^a_0, ..., \mathcal{A}^a_3) \qquad a = 1,\; ..., 8$

quarks: 4-spinor, Grassmann variables $\psi_{f,a},\;\; a = 1,2,3,\;\; f = u,d,c,s,t,b$

Gauge field theories

gauge transformation: $G(x) \equiv \mathrm{e}^{i\Lambda(x)} \in SU(3) \quad , \quad \Lambda = \sum_{\alpha=1}^{N_c^2-1} \Lambda^\alpha \frac{\lambda^\alpha}{2}$

complex $N_c \times N_c$ matrix: G

$$G^{-1} = G^\dagger \quad , \quad \det G = 1$$

$$\begin{aligned}
\psi(x) &\rightarrow G(x)\psi(x) \\
\bar\psi(x) &\rightarrow \bar\psi(x)G^{-1}(x) \\
A_\mu(x) &\rightarrow G(x)A_\mu(x)G^{-1}(x) - \frac{i}{g}G(x)\partial_\mu G^{-1}(x) \\
D_\mu(x) &\rightarrow G(x)D_\mu(x)G^{-1}(x)
\end{aligned}$$

$\Rightarrow \bar\psi(x)\psi(x)$ is gauge invariant

$\Rightarrow \bar\psi(x)D_\mu(x)\psi(x)$ is gauge invariant

$\Rightarrow \quad F'_{\mu\nu} = GF_{\mu\nu}G^{-1} \Rightarrow \ \mathrm{Tr}F_{\mu\nu}F_{\mu\nu}$ is gauge invariant

The idea of K. Wilson: lattice gauge theory

regularize QCD by introducing a discrete space-time lattice;

– violates various symmetries, but...

– minimal requirement: preserve "local" gauge invariance

Phys. Rev. D 10 (1974) 2445

parallel transporter:

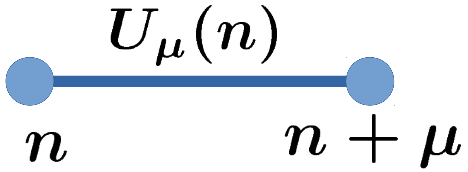
introduce gauge fields that interpolate between sites n and $n+\mu$

$$U(n,n+\mu) \equiv U_{n,\mu} \equiv U_\mu(n) = \mathrm{e}^{ig\int_x^{x+\mu} dz A_\mu(z)} = \mathrm{e}^{igaA_\mu(n)}$$

lattice spacing

gauge transformation: $U_\mu(n) \rightarrow G_n U_\mu(n) G^{-1}_{n+\mu}$

I) introduce gauge degrees of freedom (vector fields) on links



$$U_\mu(n) = \mathrm{e}^{i\theta_\mu(n)} \quad , \quad \theta_\mu(n) = gaA_\mu(n)$$

ii) replace gauge fields $A_\mu(n) \equiv A^a_\mu(n)\lambda^a/2$ by elements of the relevant symmetry group

QCD: $U_{n,\mu} \in SU(3)$

lattice discretization:

introduce gauge degrees of freedom on links,

$$U_\mu(n) \in SU(N_c) \quad , \quad U_\mu(n) \equiv \mathrm{e}^{iagA_\mu(n)} \qquad\qquad A_\mu = \sum_{\alpha=1}^{N_c^2-1} A_\mu^\alpha \frac{\lambda^\alpha}{2}$$

$N_c \times N_c$ matrix

the plaquette variable $\mathrm{Re\ Tr}\ U_{\mu\nu}(n)$

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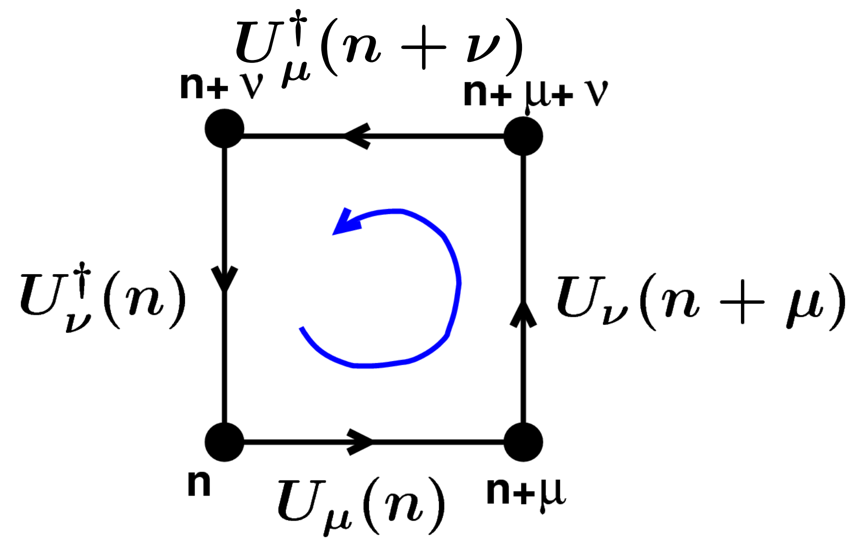

the plaquette variable $\mathrm{Re\ Tr}\ U_{\mu\nu}(n)$

$$U_{\mu\nu}(n) \equiv U_\mu(n) U_\nu(n+\mu) U_\mu^\dagger(n+\nu) U_\nu^\dagger(n)$$

- matrices do not commute
- cyclic permutation under trace

$$U_{\mu\nu}^\dagger(n) \equiv U_\nu(n) U_\mu(n+\nu) U_\nu^\dagger(n+\mu) U_\mu^\dagger(n) \quad , \quad \mathrm{Tr} U_{\mu\nu}^\dagger = (\mathrm{Tr} U_{\mu\nu})^*$$

$$U_{\mu\nu}(n) \equiv U_\mu(n) U_\nu(n+\mu) U_\mu^\dagger(n+\mu) U_\nu^\dagger(n)$$

$$= \mathrm{e}^{iagA_\mu(n)} \mathrm{e}^{iagA_\nu(n+\mu)} \mathrm{e}^{-iagA_\mu(n+\mu)} \mathrm{e}^{-iagA_\nu(n)}$$

use Baker-Hausdorff formula $\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B+[A,B]+\dots}$

$$\mathrm{e}^{iag(A_\mu(n)+A_\nu(n+\mu)) - a^2 g^2 [A_\mu(n), A_\nu(n+\mu)]}$$

use Baker-Hausdorff repeatedly

$$\Rightarrow \quad \mathrm{Re\ Tr} U_{\mu\nu}(n) = N_c - a^4 \frac{g^2}{2} \mathrm{Tr}\ F_{\mu\nu}(n) F_{\mu\nu}(n) + \mathcal{O}(a^6)$$

$$S_G = \frac{2N_c}{g^2} \sum_n \sum_{1\le\mu<\nu\le 4} \left(1 - \frac{1}{N_c} \mathrm{Re\ Tr}\ U_{\mu\nu}(n)\right) \rightarrow \int \mathrm{d}^4 x \mathcal{L}_G(x) + \mathcal{O}(a^2)$$

$$Z(\beta) = \int \prod_{n,\mu} dU_\mu(n) \mathrm{e}^{-S_G(U)} \quad , \quad \beta \equiv \frac{2N_c}{g^2}$$

[not the inverse temperature!!]

$$U_{\mu\nu}(n) \equiv U_\mu(n)U_\nu(n+\mu)U^\dagger_\mu(n+\mu)U^\dagger_\nu(n)$$

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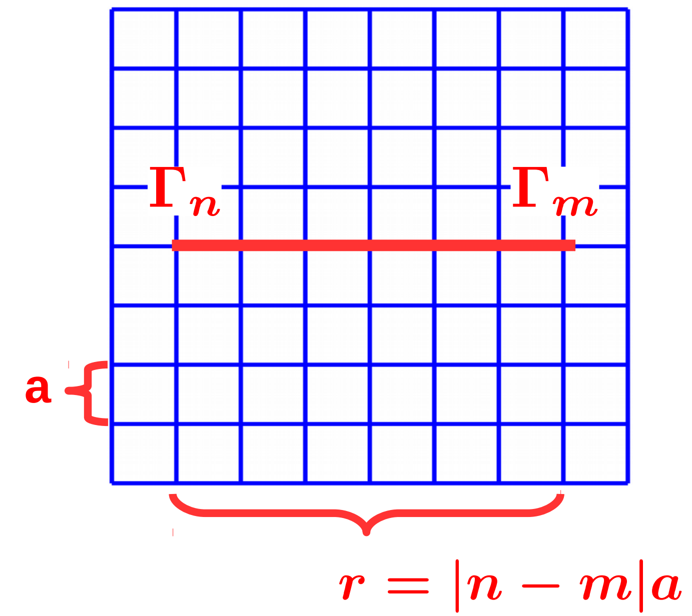
lattice spacing "a" disappeared"

Continuum limit

long distance behavior of 2-point function:



$$\langle \Gamma_n \Gamma_m \rangle \sim e^{-M|n-m|}$$
$$\equiv e^{-|n-m|/\xi}$$

ξ : correlation length (statistical physics)

$M \equiv \xi^{-1}$:
mass of a scalar particle in lattice units; $M \equiv m_c a$

continuum limit: $a \rightarrow 0$, implies $M \rightarrow 0$ for any fixed (physical) mass m_c

$$M \rightarrow 0 \Leftrightarrow \xi \rightarrow \infty$$

$\xi \rightarrow \infty$: divergent correlation length $\Rightarrow$ (2nd order) phase transition in statistical physics terminology

behavior of ξ defines the lattice spacing $\xi \equiv c/a$

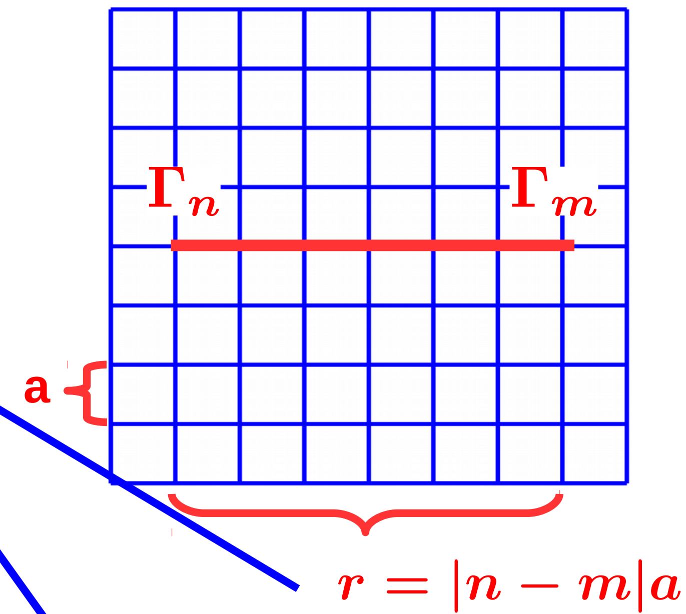
Continuum limit

long distance behavior of 2-point function:

$$\langle \Gamma_n \Gamma_m \rangle \sim e^{-M|n-m|} \sim e^{-m_c r}$$

$$\equiv e^{-|n-m|/\xi}$$

for instance:
$\Gamma_n = \bar{\psi}_n \gamma_5 \psi_n$



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The continuum limit of lattice regularized QCD

- at present we consider only SU(Nc) gauge theories

- the only free parameter is the gauge coupling g or $\beta = 2N_c/g^2$

$$\Rightarrow \quad g^2 \equiv g^2(a)$$

Continuum limit: $a \rightarrow 0 \quad \Leftrightarrow \quad \xi \rightarrow \infty$ divergent correlation length

The gauge coupling g^2 needs to approach a critical point g_*^2
at which the 'correlation lengths', defined through 2-point functions, diverge, e.g.

$$G_2(\vec{n}) \equiv \langle \Gamma(0)\Gamma^\dagger(\vec{n})\rangle \sim \mathrm{e}^{-|\vec{n}|/\hat{\xi}_\Gamma}$$

$$\hat{\xi}_\Gamma \equiv \hat{\xi}_\Gamma(\beta) \equiv \hat{\xi}_\Gamma(g^2) = \xi_\Gamma/a$$

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$$\hat{\xi}_\Gamma \equiv \hat{\xi}_\Gamma(\beta) \equiv \hat{\xi}_\Gamma(g^2) = \xi_\Gamma/a$$

<= often used notation, but not quite correct

Continuum limit: physical observables, $\Gamma = m_G,\ \sigma,\ r_1,\ \epsilon,\ ...$
do not depend on the lattice spacing (cut-off)

$$a\frac{\mathrm{d}}{\mathrm{d}a}\Gamma = 0 \qquad (1)$$

- observables calculated in a regularized QFT depend on the cut-off, e.g. they depend on the lattice spacing

- observables calculated on the lattice are dimensionless, i.e. they are calculated in appropriate units of the lattice spacing

$$\hat{\Gamma} \equiv a^{d_\Gamma}\Gamma$$

$$\hat{\Gamma} \equiv \hat{\Gamma}(g) \Leftrightarrow \Gamma = a^{-d_\Gamma}\hat{\Gamma} \equiv \Gamma(a,g)$$

(1) now reads: $$a\frac{\mathrm{d}}{\mathrm{d}a}\Gamma = \left(a\frac{\partial}{\partial a} - \beta(g)\frac{\partial}{\partial g}\right)\Gamma = 0$$

with $$\beta(g) = -a\frac{\mathrm{d}g}{\mathrm{d}a}$$ β -function

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renormalization group equation

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The β-function: $$\beta(g) = -a\frac{\mathrm{d}g}{\mathrm{d}a}$$

$$\Leftrightarrow \frac{1}{a}\mathrm{d}a = -\frac{1}{\beta(g)}\mathrm{d}g$$

$$\int_a^{a_0} \frac{1}{a'}\mathrm{d}a' = -\int_g^{g_0} \frac{1}{\beta(g')}\mathrm{d}g'$$

The β-function: $\beta(g) = -a\frac{\mathrm{d}g}{\mathrm{d}a}$

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$$\int_a^{a_0} \frac{1}{a'}\mathrm{d}a' = -\int_g^{g_0} \frac{1}{\beta(g')}\mathrm{d}g'$$

$\ln(a_0/a)$

$$\Rightarrow \frac{a}{a_0} = \mathrm{e}^{\int_g^{g_0} \frac{1}{\beta(g')}\mathrm{d}g'} \equiv R(g, g_0)$$

The β-function

- controls the variation of the lattice spacing as function of the gauge coupling

- provides a solution to the renormalization group equation

The β-function: $$\beta(g) = -a\frac{\mathrm{d}g}{\mathrm{d}a}$$

$$\Leftrightarrow \frac{1}{a}\mathrm{d}a = -\frac{1}{\beta(g)}\mathrm{d}g$$

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$$\Rightarrow \frac{a}{a_0} = \mathrm{e}^{\int_g^{g_0} \frac{1}{\beta(g')}\mathrm{d}g'} \equiv R(g, g_0)$$

continuum limit: $$a \to 0 \quad \Leftrightarrow \quad \int_{g_*}^{g_0} \mathrm{d}g' .... \to -\infty$$

$$\Leftrightarrow \quad \beta(g_*) = 0$$

β-function needs to have a zero (fixpoint)

Asymptotic Freedom

fixpoints: $\beta(g_*) = 0$

$$\beta(g) = (g - g_*)^r \; b_0 \quad , \quad r \ge 1$$

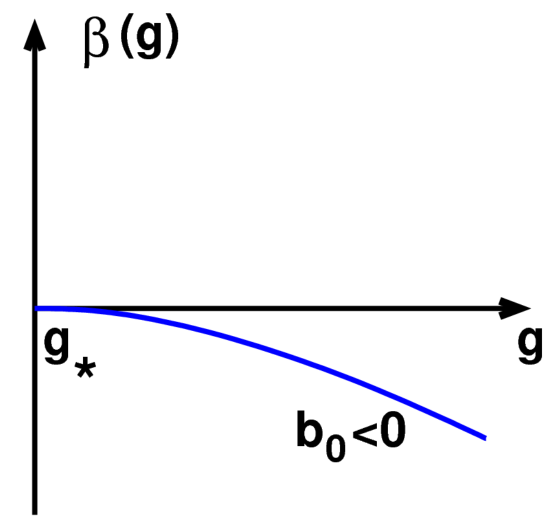
a special case: $g_* = 0 \quad \Rightarrow \quad \frac{a}{a_0} = e^{\frac{1}{b_0(r-1)}\left(g^{-(r-1)} - g_0^{-(r-1)}\right)}$

$$a\Lambda_L = e^{-\frac{1}{|b_0|(r-1)} \frac{1}{g^{(r-1)}}}$$

$$g \to 0 \quad \Leftrightarrow \quad a \to 0$$

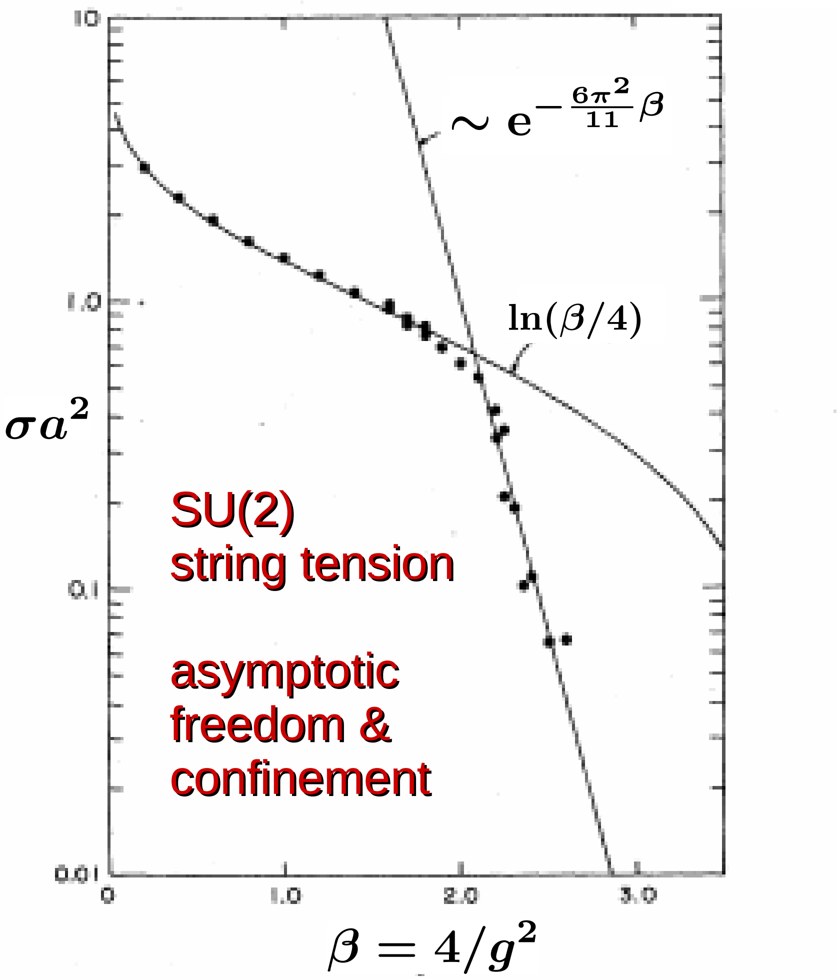
QCD beta-function $r = 3 \; , \; b_0 < 0$

$$\beta(g) = -\left(\frac{11N_c}{3} - \frac{2n_f}{3}\right)\frac{g^3}{16\pi^2} + \mathcal{O}(g^5)$$



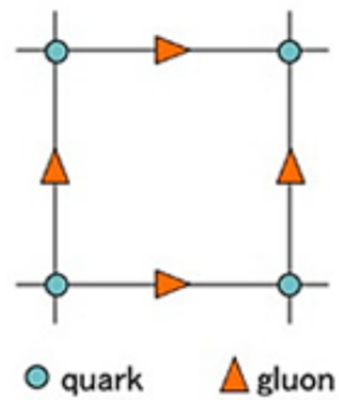
Asymptotic Freedom and the Heavy Quark Potential

heavy quark potential: $V(r) = -\frac{\alpha}{r} + \sigma r$



M. Creutz,
Phys. Rev. D21 (1980) 2308

Monte Carlo simulation 1979



Mike Creutz

Monte Carlo Simulations

Observables = Expectation values

$$Z = \int \mathcal{D}\mathcal{A}\mathcal{D}\psi\mathcal{D}\bar{\psi}\, \mathrm{e}^{-S_E} \qquad \langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}\mathcal{A}\mathcal{D}\psi\mathcal{D}\bar{\psi}\, \mathcal{O}\mathrm{e}^{-S_E}$$

integrate out fermions:

$$Z = \int \mathcal{D}\mathcal{A}\mathcal{D}\psi\mathcal{D}\bar{\psi}\, \mathrm{e}^{-S_E} = \int \mathcal{D}\mathcal{A} \prod_f \det M(m_f) \mathrm{e}^{-S_G} = \int \mathcal{D}\mathcal{A}\, \mathrm{e}^{-S}$$

$$S = S_G - \sum_f \mathrm{Tr}\, \ln M(m_f)$$

distribute gauge field configurations $\{\mathcal{A}^{(i)}\}_{i=1}^{N_{tot}}$ according to the probability distribution $P(\{\mathcal{A}\}) = Z^{-1}\mathrm{e}^{-S(\{\mathcal{A}\})}$

Monte Carlo algorithms (detailed balance) generates Markov chain

calculate expectation values: $\langle \mathcal{O} \rangle = \lim_{N_{tot}\to\infty} \frac{1}{N_{tot}} \sum_{i=1}^{N_{tot}} \mathcal{O}(\{\mathcal{A}^{(i)}\})$

Different regularization schemes for fermions (and gauge sector)

– staggered fermions → doublers
– Wilson fermions → explict chiral symmetry breaking

– Domain Wall Fermions → 5-dimensional formulation, "almost" exact chiral symmetry, "residual mass"

– Overlap fermions → chiral fermion formulation (obeys Ginsparg-Wilson relation)

chiral fermions are computationally demanding

overlap $\sim$ 100 x staggered

Improved actions

try to eliminate $\mathcal{O}(a^2)$ discretization errors in the continuum limit

– pure gauge sector: Symanzik improvement, e.g. at 6-link terms to action
– fermion sector: higher order discretization schemes, e.g. add three link terms: naïve staggered → Naik action → Highly improved staggered quarks (HISQ)
Wilson fermions have $\mathcal{O}(a)$ errors, add "clover-term" to arrive at $\mathcal{O}(a^2)$