Relativistic Heavy Ion Collisions: Theory

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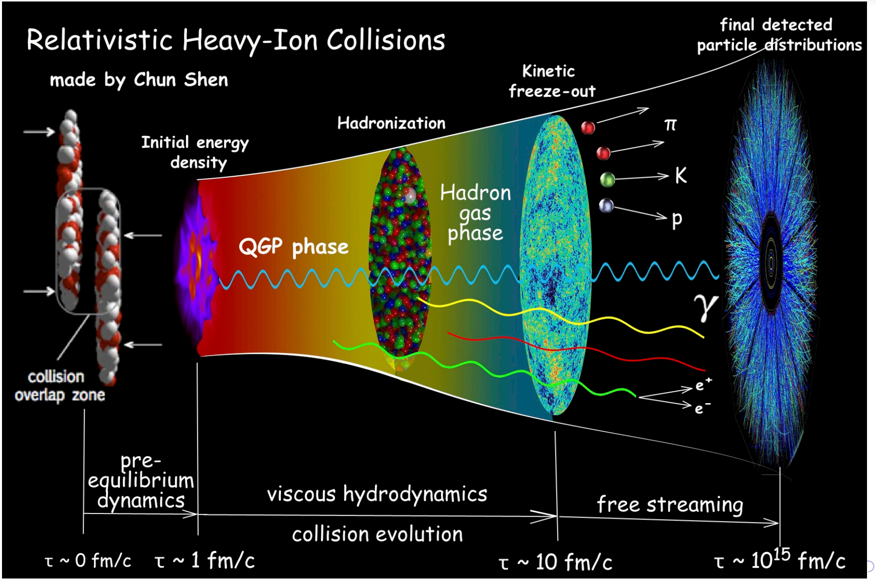
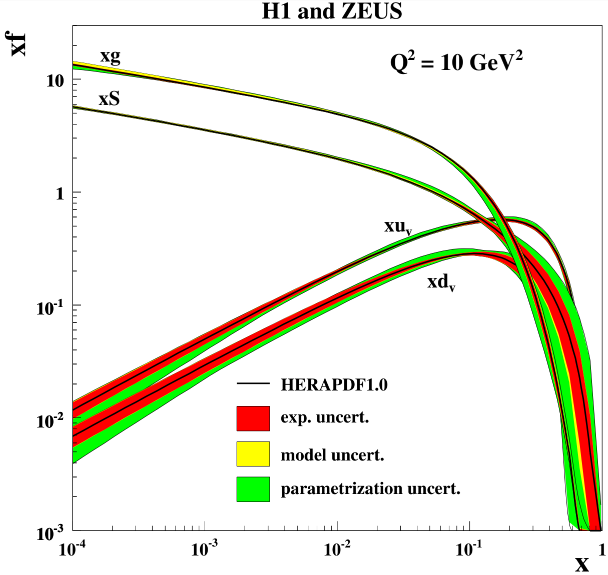
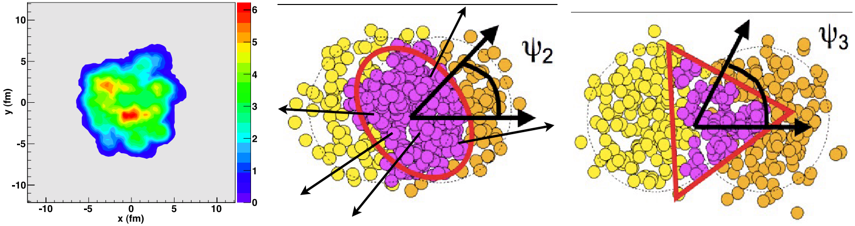
The Little Bang (Credit: Chun Shen/Paul Sorensen)



The action is where the glue is!



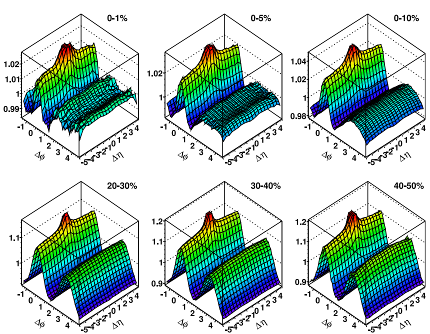
Event-by-event shape and flow fluctuations rule!



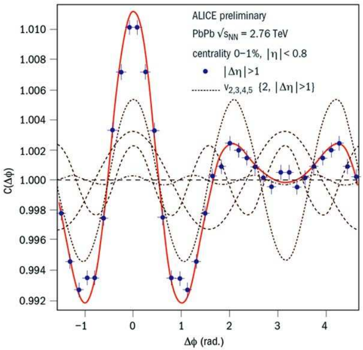
- Each event has a different initial shape and density distribution, characterized by different set of harmonic eccentricity coefficients ε_n
- Each event develops its individual hydrodynamic flow, characterized by a set of harmonic flow coefficients v_n and flow angles ψ_n
- At small impact parameters fluctuations ("hot spots") dominate over geometric overlap effects (Alver & Roland, PRC81 (2010) 054905; Qin, Petersen, Bass, Müller, PRC82 (2010) 064903)

Panta rhei! "soft ridge" = "Mach cone" = flow!

ATLAS (J. Jia), Quark Matter 2011



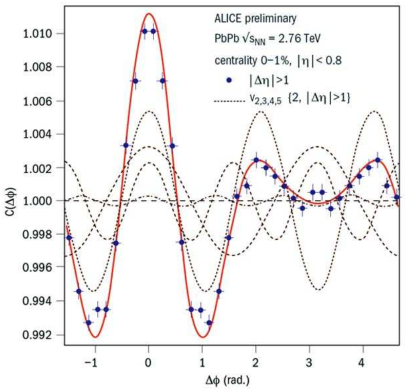
ALICE (J. Grosse-Oetringhaus), QM11



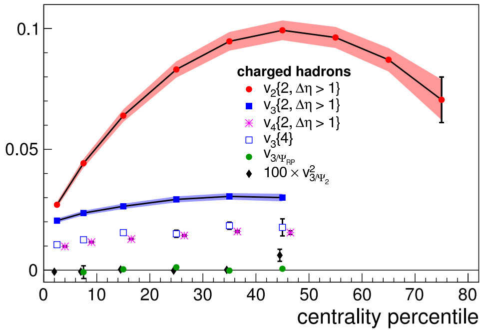
- anisotropic flow coefficients v_n and flow angles ψ_n correlated over large rapidity range!
  M. Luzum, PLB 696 (2011) 499: All long-range rapidity correlations seen at RHIC are consistent with being entirely generated by hydrodynamic flow.

- in the 1% most central collisions $v_3 > v_2$
  $\Longrightarrow$ prominent "Mach cone"-like structure!
  $\Longrightarrow$ event-by-event eccentricity fluctuations dominate!

Event-by-event shape and flow fluctuations rule!



ALICE (A. Bilandzic) Quark Matter 2011



- in the 1% most central collisions $v_3 > v_2 \Longrightarrow$ prominent "Mach cone"-like structure!
- triangular flow angle uncorrelated with reaction plane and elliptic flow angles $\Longrightarrow$ due to event-by-event eccentricity fluctuations which dominate the anisotropic flows in the most central collisions

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Kinetic theory vs. hydrodynamics

Both simultaneously valid if weakly coupled and small pressure gradients.

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Boltzmann Equation in Relaxation Time Approximation (RTA):

$$p^\mu \partial_\mu f(x,p) = C(x,p) = \frac{p\cdot u(x)}{\tau_{\rm rel}(x)}\Big(f_{\rm eq}(x,p) - f(x,p)\Big)$$

For conformal systems $\tau_{\rm rel}(x) = c/T(x) = 5\eta/(sT) \equiv 5\bar{\eta}/T(x)$.

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Macroscopic currents:

$$j^\mu(x) = \int_p p^\mu\, f(x,p) \equiv \langle p^\mu \rangle; \quad T^{\mu\nu}(x) = \int_p p^\mu\, p^\nu\, f(x,p) \equiv \langle p^\mu p^\nu \rangle$$

where $$\int_p \cdots \equiv \frac{g}{(2\pi)^3}\int \frac{d^3p}{E_p} \cdots \equiv \langle \dots \rangle$$

Hydrodynamics for strongly anisotropic expansion (I)

Account for large viscous flows by including their effect already at leading order in the Chapman-Enskog expansion:

Expand the solution $f(x, p)$ of the Boltzmann equation as

$$f(x,p) = f_0(x,p) + \delta f(x,p) \qquad \left(\left|\delta f/f_0\right| \ll 1\right)$$

where f_0 is parametrized through macroscopic observables as

$$f_0(x,p) = f_0\left(\frac{\sqrt{p_\mu \Xi^{\mu\nu}(x) p_\nu} - \tilde{\mu}(x)}{\tilde{T}(x)}\right)$$

where $\Xi^{\mu\nu}(x) = u^\mu(x)u^\nu(x) - \Phi(x)\Delta^{\mu\nu}(x) + \xi^{\mu\nu}(x).$

$u^\mu(x)$ defines the local fluid rest frame (LRF).
$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is the spatial projector in the LRF.
$\tilde{T}(x)$, $\tilde{\mu}(x)$ are the effective temperature and chem. potential in the LRF.
$\Phi(x)$ accounts for bulk viscous effects in expanding systems.
$\xi^{\mu\nu}(x)$ describes deviations from local momentum isotropy in anisotropically expanding systems due to shear viscosity.

Hydrodynamics for strongly anisotropic expansion (II)

$u^\mu(x)$, $\tilde{T}(x)$, $\tilde{\mu}(x)$ are fixed by the Landau matching conditions:

$$T^\mu_{\ \nu} u^\nu = \mathcal{E}(\tilde{T}, \tilde{\mu}; \xi, \Phi) u^\mu; \qquad \Big\langle u{\cdot}p \Big\rangle_{\delta f} = \Big\langle (u{\cdot}p)^2 \Big\rangle_{\delta f} = 0$$

$\mathcal{E}$ is the LRF energy density. We introduce the true local temperature $T(\tilde{T}, \tilde{\mu}; \xi, \Phi)$ and chemical potential $\mu(\tilde{T}, \tilde{\mu}; \xi, \Phi)$ by demanding $\mathcal{E}(\tilde{T}, \tilde{\mu}; \xi, \Phi) = \mathcal{E}_{\rm eq}(T, \mu)$ and $\mathcal{N}(\tilde{T}, \tilde{\mu}; \xi, \Phi) \equiv \langle u{\cdot}p \rangle_{f_0} = \mathcal{N}_{\rm eq}(T, \mu)$.
Writing

$$T^{\mu\nu} = T_0^{\mu\nu} + \delta T^{\mu\nu} \equiv T_0^{\mu\nu} + \Pi^{\mu\nu}, \qquad j^\mu = j_0^\mu + \delta j^\mu \equiv j_0^\mu + V^\mu,$$

the conservation laws

$$\partial_\mu T^{\mu\nu}(x) = 0, \qquad \partial_\mu j^\mu(x) = \frac{\mathcal{N}(x) - \mathcal{N}_{\rm eq}(x)}{\tau_{\rm rel}(x)}$$

are sufficient to determine $u^\mu(x)$, $T(x)$, $\mu(x)$, but not the dissipative corrections $\xi^{\mu\nu}$, Φ, $\Pi^{\mu\nu}$, and V^μ whose evolution is controlled by microscopic physics.

Hydrodynamics for strongly anisotropic expansion (III)

Different hydrodynamic approaches can be characterized by the different assumptions they make about the dissipative corrections and/or the different approximations they use to derive their dynamics from the underlying Boltzmann equation:

- **Ideal hydro:** local momentum isotropy ($\xi^{\mu\nu} = 0$), $\Phi = \Pi^{\mu\nu} = V^{\mu} = 0$.

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- **Anisotropic hydrodynamics (aHydro):** allows for leading-order local momentum anisotropy ($\xi^{\mu\nu}$, $\Phi \neq 0$), evolved according to equations obtained from low-order moments of BE, but ignores residual dissipative flows: $\Pi^{\mu\nu} = V^\mu = 0$.

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- **Viscous anisotropic hydrodynamics (vaHydro):** improved **aHydro** that additionally evolves residual dissipative flows $\Pi^{\mu\nu}$, V^μ with **IS** or **DNMR theory**.

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BE for systems with highly symmetric flows: I. Bjorken flow

- Longitudinal boost invariance, transverse homogeneity ("physics on the light cone", no transverse flow) $\Longrightarrow$ $\boldsymbol{u^\mu = (1,0,0,0)}$ in Milne coordinates (τ, r, ϕ, η) where $\tau = (t^2 - z^2)^{1/2}$ and $\eta = \frac{1}{2}\ln[(t-z)/(t+z)]$ $\Longrightarrow$ $\boldsymbol{v_z = z/t}$

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- Symmetry restricts possible dependence of distribution function $f(x,p)$ (Baym '84, Florkowski et al. '13, '14):

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- **Solution:**

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where $\qquad D(\tau_2, \tau_1) = \exp\left(-\int_{\tau_1}^{\tau_2} \frac{d\tau''}{\tau_{\rm rel}(\tau'')} \right).$

BE for systems with highly symmetric flows: II. Gubser flow

- Longitudinal boost invariance, azimuthally symmetric radial dependence ("physics on the light cone" with azimuthally symmetric transverse flow)
(Gubser '10, Gubser & Yarom '11)
$\Longrightarrow$ $\boldsymbol{u^\mu = (1,0,0,0)}$ in de Sitter coordinates (ρ,θ,ϕ,η) where
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- With $T(\tau, r) = \hat{T}(\rho(\tau, r))/\tau$ RTA BE simplifies to the ODE

$$\frac{\partial}{\partial\rho} f(\rho; \hat{p}_\Omega^2, \hat{p}_\varsigma) = -\frac{\hat{T}(\rho)}{c}\left[f\left(\rho; \hat{p}_\Omega^2, \hat{p}_\varsigma\right) - f_{\rm eq}\left(\hat{p}^\rho/\hat{T}(\rho)\right)\right].$$

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- **Solution:**
  $f(\rho;\hat{p}_\Omega^2,w) = D(\rho,\rho_0) f_0(\hat{p}_\Omega^2,w) + \frac{1}{c}\int_{\rho_0}^{\rho} d\rho'\,\hat{T}(\rho')\,D(\rho,\rho')\,f_{\rm eq}(\rho';\hat{p}_\Omega^2,w)$

Hydrodynamic equations for systems with Gubser flow:

- The exact solution for f can be worked out for any "initial" condition $f_0(\hat{p}_\Omega^2, w) \equiv f(\rho_0; \hat{p}_\Omega^2, w)$. Here I use equilibrium initial conditions, $f_0 = f_{\rm eq}$.

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- This exact solution of the BE can be compared to solutions of the various hydrodynamic equations in de Sitter coordinates, using identical initial conditions.
  - **Ideal:** $\hat{T}_{\rm ideal}(\rho) = \frac{\hat{T}_0}{\cosh^{2/3}(\rho)}$
  - **NS:** $\frac{1}{\hat{T}}\frac{d\hat{T}}{d\rho} + \frac{2}{3}\tanh\rho = \frac{1}{3}\bar{\pi}^\eta_\eta(\rho)\tanh\rho$ (viscous T-evolution)
    with $\bar{\pi}^\eta_\eta \equiv \hat{\pi}^\eta_\eta/(\hat{T}\hat{s})$ and $\hat{\pi}^{\eta\eta}_{NS} = \frac{4}{3}\hat{\eta}\tanh\rho$ where $\frac{\hat{\eta}}{\hat{s}} \equiv \bar{\eta} = \frac{1}{5}\hat{T}\hat{\tau}_{\rm rel}$
  - **IS:** $\frac{d\bar{\pi}^\eta_\eta}{d\rho} + \frac{4}{3}\left(\bar{\pi}^\eta_\eta\right)^2\tanh\rho + \frac{\bar{\pi}^\eta_\eta}{\hat{\tau}_{\rm rel}} = \frac{4}{15}\tanh\rho$
  - **DNMR:** $\frac{d\bar{\pi}^\eta_\eta}{d\rho} + \frac{4}{3}\left(\bar{\pi}^\eta_\eta\right)^2\tanh\rho + \frac{\bar{\pi}^\eta_\eta}{\hat{\tau}_{\rm rel}} = \frac{4}{15}\tanh\rho + \frac{10}{21}\bar{\pi}^\eta_\eta\tanh\rho$
  - **aHydro:** see M. Nopoush et al., PRD 91 (2015) 045007
  - **vaHydro:** see M. Martinez et al., PRC95 (2017) 054907

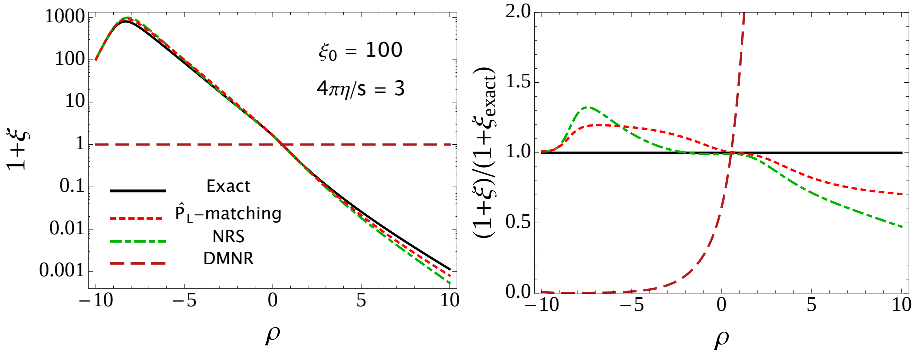
Exact BE vs. hydrodynamic approximations: Gubser flow

Optimal evolution of the momentum deformation parameter ξ?

- **"Standard" viscous hydrodynamics (IS or DNMR)**:
  expansion around local equilibrium $\Longrightarrow \boldsymbol{\xi \equiv 0}$
- **Anisotropic hydrodynamics:**
  expansion around a locally momentum-anisotropic state $\Longrightarrow \boldsymbol{\xi \neq 0}$
  - ***P_L*-matching** (Tinti 2015; Molnar, Niemi, Rischke, 2016):
    Additional Landau matching condition that matches ξ evolution to that of the longitudinal pressure $P_L \Longrightarrow$ no $\delta\tilde{f}$ corrections to P_L.
    In this case ξ can be eliminated, and the evolution equations can be written entirely in terms of macroscopic variables, as in standard viscous hydrodynamics
  - **NSR approach** (Nopoush, Strickland, Ryblewski 2015):
    obtain ξ evolution equation from second moments of the BE
    $\Longrightarrow P_L$ evolution not fully captured by ξ evolution.
  - **NLO-NSR approach** (Martinez, McNelis, UH 2017):
    Same ξ evolution but includes residual $\delta\tilde{f}$ contribution to P_L
    This captures the missing part of the pressure anisotropy.

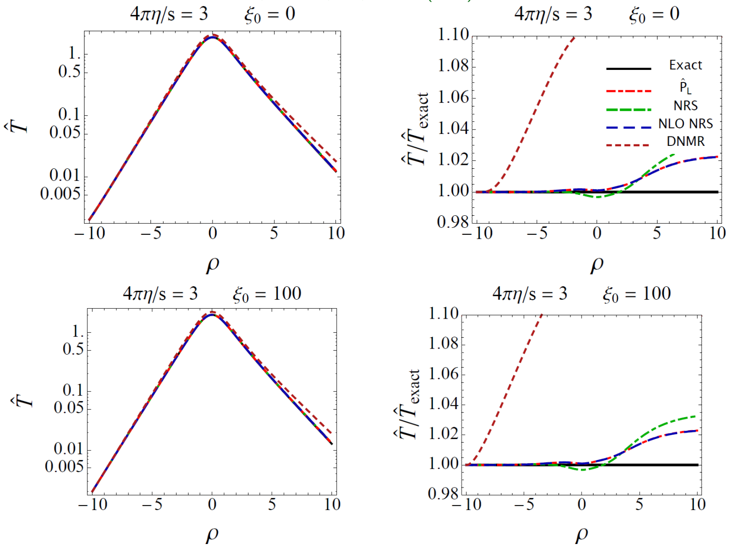
Exact BE vs. hydrodynamic approximations: Gubser flow

Martinez, McNelis, UH, PRC95 (2017) 054907



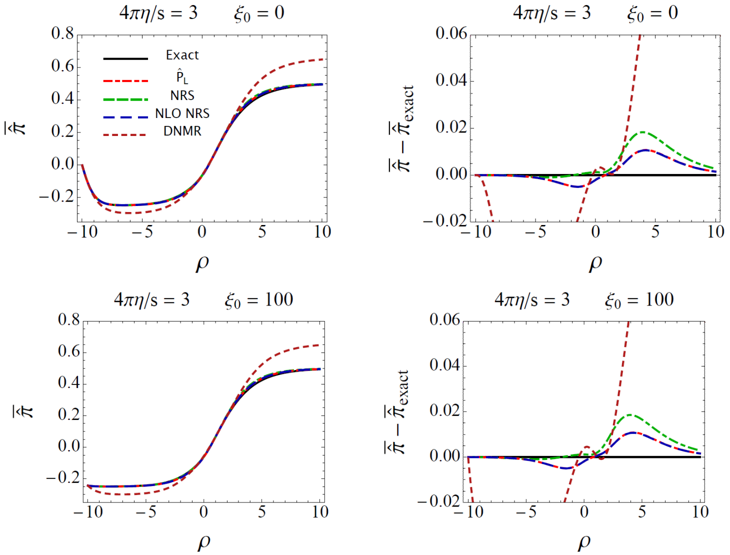
Exact BE vs. hydrodynamic approximations: Gubser flow

Martinez, McNelis, UH, PRC95 (2017) 054907



Exact BE vs. hydrodynamic approximations: Gubser flow

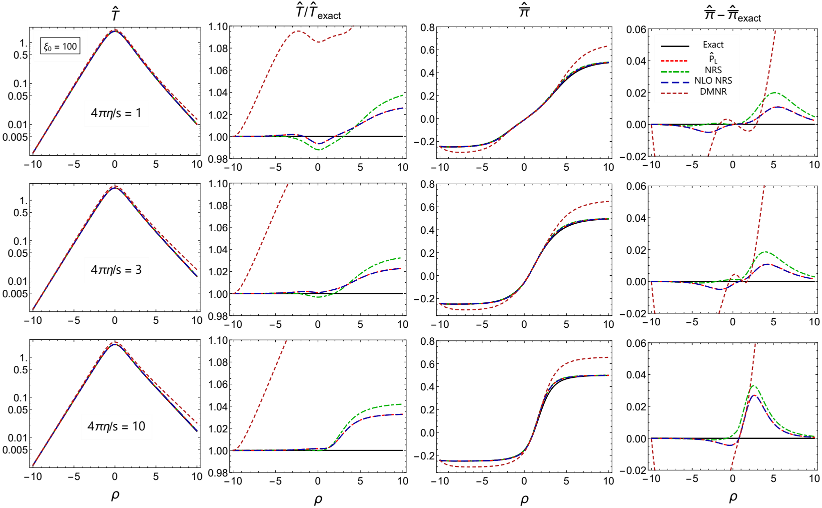
Martinez, McNelis, UH, PRC95 (2017) 054907



$\bar{\hat{\pi}} \equiv \hat{\pi}/(4\hat{P})$

Exact BE vs. hydrodynamic approximations: Gubser flow

Martinez, McNelis, UH, PRC95 (2017) 054907

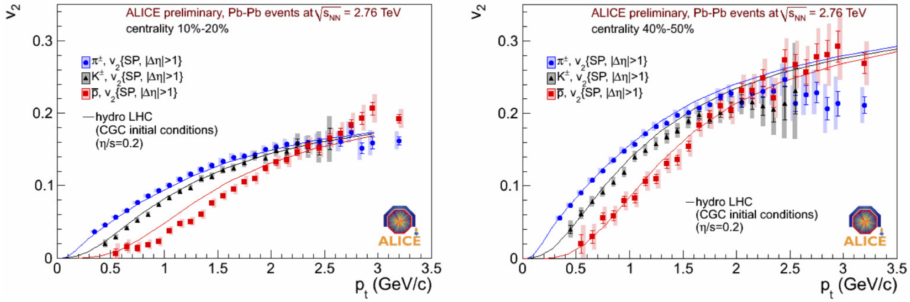


Relativistic Heavy Ion Collisions: Theory

Towards a really predictive theory of relativistic heavy-ion collision dynamics

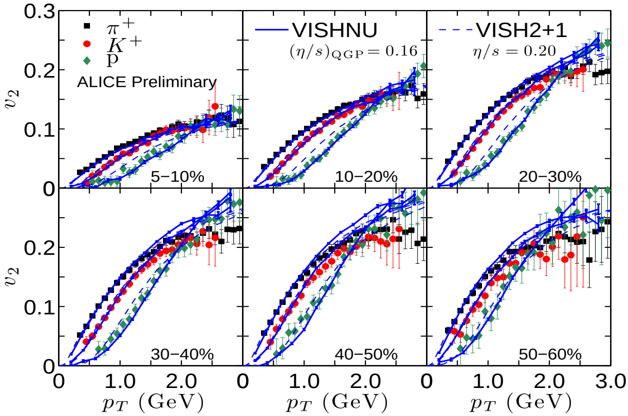
After tuning initial conditions and viscosity at RHIC to obtain a good description of all soft hadron data simultaneously (Song et al. 2010) we successfully predicted the first LHC spectra and elliptic flow measurements:

ALICE, Quark Matter 2011 (VISH2+1 prediction: Shen et al., PRC84 (2011) 044903)



Hybrid (hydro+cascade) approaches work even better:

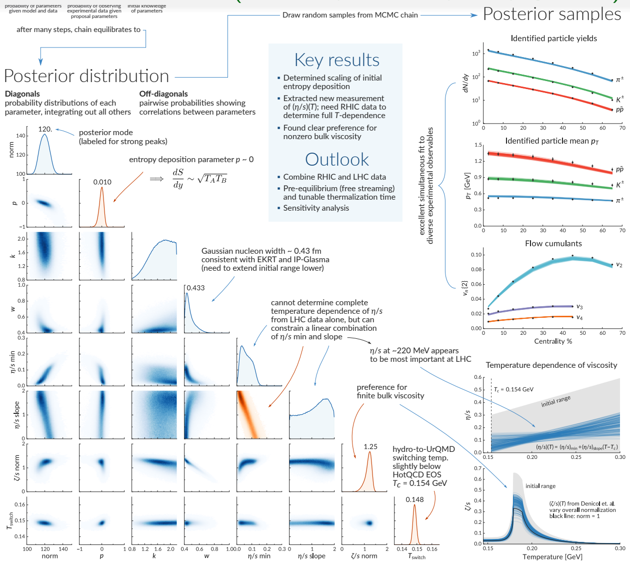
Data: ALICE, preliminary (Snellings, Krzewicki, Quark Matter 2011)
Dashed lines: Shen et al., PRC84 (2011) 044903 (VISH2+1, MC-KLN, $(\eta/s)_{\rm QGP}$=0.2)
Solid lines: Song, Shen, UH 2011 (VISHNU, MC-KLN, $(\eta/s)_{\rm QGP}$=0.16)



VISHNU yields correct magnitude and centrality dependence of $v_2(p_T)$ for pions, kaons **and protons**!

The state of the art

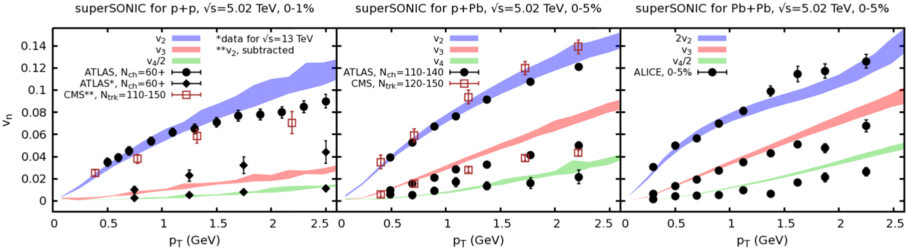
(Bernhard, Moreland, Bass, QM2015)



Relativistic Heavy Ion Collisions: Theory

Flow in Pb+Pb, p+Pb and even p+p at the LHC!

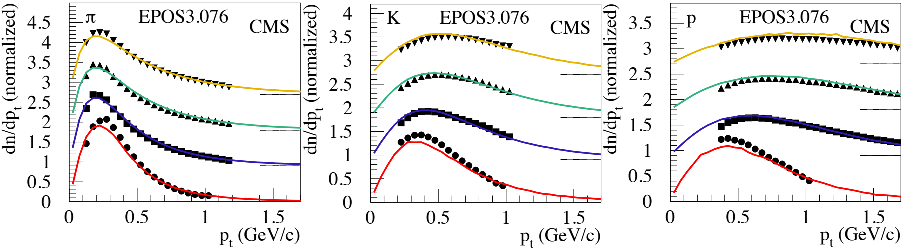
R.D. Weller, P. Romatschke, arXiv:1701.07145



Requires fluctuating proton substructure (gluon clouds clustered around valence quarks (K. Welsh et al. PRC94 (2016) 024919))

Radial flow in pp collisions at the LHC

Werner, Guiot, Karpenko, Pierog (EPOS3), 1312.1233;
Data: CMS Collaboration (8, 84, 160, 235 charged tracks)

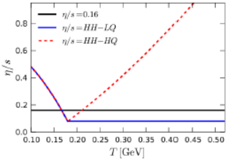


Elliptic flow (double ridge) discovered in high-multiplicity pp by CMS at 7 TeV (and confirmed by ATLAS at 13 TeV) also reproduced by EPOS.

Validity of viscous hydro: Knudsen number check

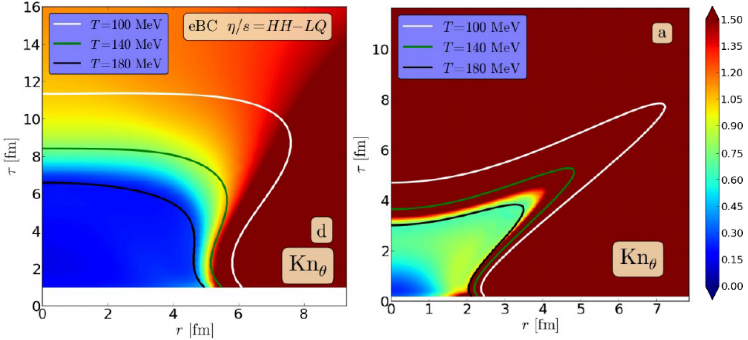
Niemi & Denicol, arXiv:1404.7327

$$\mathrm{Kn} = \tau_{\mathrm{micro}}\,\theta = \tau_{\mathrm{micro}}/\tau_{\mathrm{macro}}$$



Pb+Pb

p+Pb

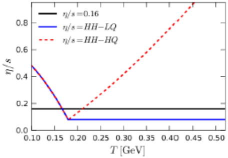


Earlier freeze-out in p+A than A+A

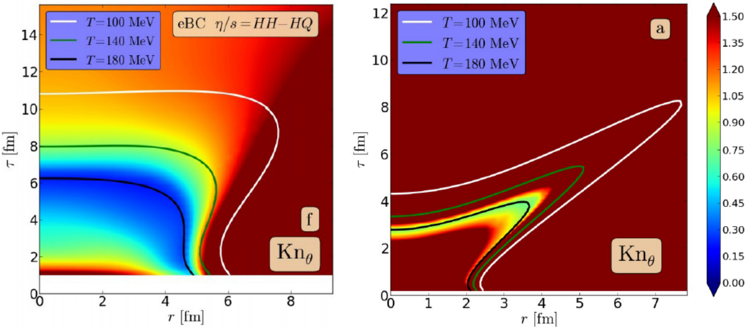
Validity of viscous hydro: Knudsen number check

Niemi & Denicol, arXiv:1404.7327

$$\mathrm{Kn} = \tau_{\mathrm{micro}}\,\theta = \tau_{\mathrm{micro}}/\tau_{\mathrm{macro}}$$



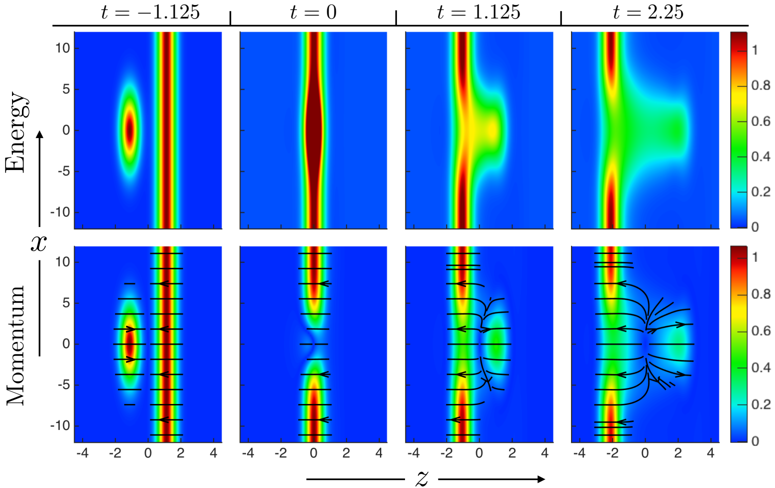
Pb+Pb **p+Pb**



Strong linear rise of η/s above T_c testing the limits of applicability of hydrodynamics in p+A collisions?

Validity of viscous hydro: Exact solution at ∞ coupling

Chesler, arXiv:1506.02209, colliding shock waves in AdS_5 for p+A



Validity of viscous hydro: Exact solution at ∞ coupling

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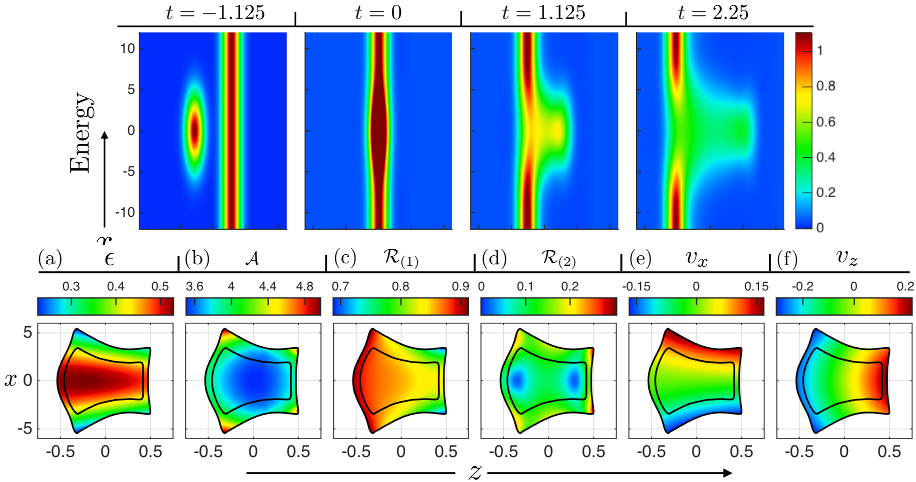


First-order terms in Re^{-1} large, but second-order terms small almost everywhere!

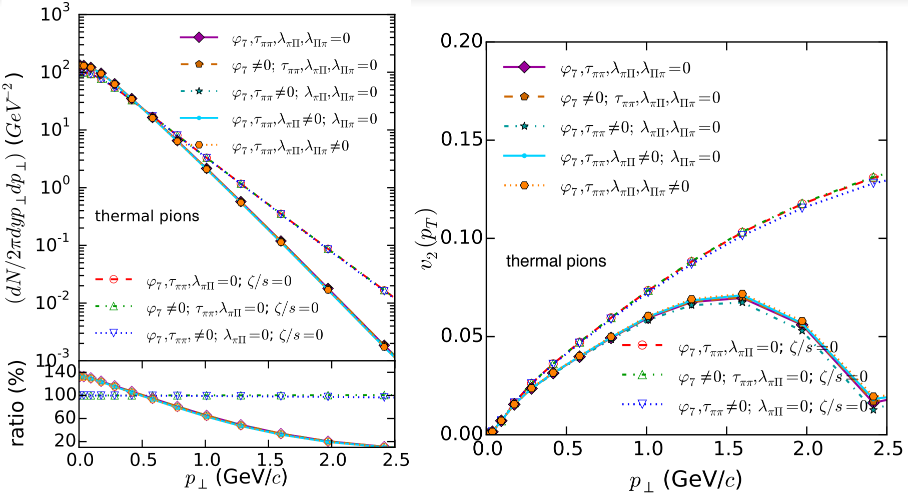
Importance of second-order terms in Kn and Re^{-1} in A+A:

$$\Delta^{\mu\alpha}\Delta^{\nu\beta}u^\lambda d_\lambda \pi_{\alpha\beta} = -\frac{1}{\tau_\pi}(\pi^{\mu\nu} - 2\eta\sigma^{\mu\nu}) - \frac{\delta_{\pi\pi}\pi^{\mu\nu}\theta}{\tau_\pi} + \frac{\varphi_7}{\tau_\pi}\pi_\alpha^{\langle\mu}\pi^{\nu\rangle\alpha} - \frac{\tau_{\pi\pi}}{\tau_\pi}\pi_\alpha^{\langle\mu}\sigma^{\nu\rangle\alpha} + \frac{\lambda_{\pi\Pi}}{\tau_\pi}\Pi\sigma^{\mu\nu},$$

$$u^\lambda d_\lambda \Pi = -\frac{1}{\tau_\Pi}(\Pi + \zeta\theta) - \frac{\delta_{\Pi\Pi}}{\tau_\Pi}\Pi\theta + \frac{\lambda_{\Pi\pi}}{\tau_\Pi}\pi^{\mu\nu}\sigma_{\mu\nu},$$

with transport coefficients from Boltzmann equation for massless Boltzmann gas.

Importance of second-order terms in Kn and Re^{-1} in A+A:



Bulk viscosity matters

Non-linear second-order terms make almost no difference

Relativistic Heavy Ion Collisions: Theory

Summary

- Viscous relativistic hydrodynamics provides a **robust, reliable, efficient and accurate** description of QGP evolution in heavy-ion collisions.
- It is valid even when the expansion is fast and highly anisotropic, causing large local momentum anisotropies $\Longrightarrow$ **local thermalization not strictly required**.
- While first-order viscous corrections are large in nuclear collisions, especially in small systems, they can be handled efficiently in an **optimized anisotropic hydrodynamic approach** that accounts for local momentum anisotropies at leading order; residual dissipative flows remain small.
- **New exact solutions of the Boltzmann equation** enable powerful tests of the efficiency and accuracy of various hydrodynamic expansion schemes, providing strong support for the **validity and robustness** of second-order viscous hydrodynamics (especially their anisotropic variants).