NNPSS 2017: Relativistic Heavy Ion Collisions: Theory

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1.) The Little Bang
show Chun Shen's sketch and explain it. FOCUS ON THE PHYSICS OF THE BULK!

2.) Event-by-event shape and flow fluctuations rule!

$$\mathcal{E}_n \equiv \varepsilon_n e^{in\Phi_n} = -\frac{\int r\,dr\,d\varphi\; r^n e^{in\varphi}\, e(r,\varphi)}{\int r\,dr\,d\varphi\; r^n\, e(r,\varphi)} \equiv \langle r^n e^{in\varphi}\rangle_e$$

for $n \geq 2$

$$\mathcal{E}_1 \equiv \varepsilon_1 e^{i\Phi_1} \equiv -\frac{\int r\,dr\,d\varphi\; r^3 e^{i\varphi}\, e(r,\varphi)}{\int r\,dr\,d\varphi\; r^3\, e(r,\varphi)}$$

"eccentricity coefficients" ε_n and "participant plane angles" Φ_n

$$V_n(p_T,y) \equiv v_n(p_T,y)\, e^{in\Psi_n(p_T,y)}$$

$$\equiv \lim_{N\to\infty}\left(\frac{1}{N}\sum_{k=1}^{N} e^{in\varphi_k}\right)_{p_T,y} \equiv \lim_{N\to\infty} Q_n(p_T,y)$$

↑ ∞ multiplicity per event

$$\equiv \frac{\int d\varphi_p\, e^{in\varphi_p}\frac{dN}{dy\,p_T\,dp_T\,d\varphi_p}}{\int d\varphi_p\,\frac{dN}{dy\,p_T\,dp_T\,d\varphi_p}}$$

$$V_n \equiv v_n e^{in\Psi_n} = \lim_{N\to\infty} Q_n = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} e^{in\varphi_k} = \frac{\int p_T\,dp_T\,dy\,d\varphi_p\, e^{in\varphi_p}\frac{dN}{dy\,p_T\,dp_T\,d\varphi_p}}{\int p_T\,dp_T\,dy\,d\varphi_p\,\frac{dN}{dy\,p_T\,dp_T\,d\varphi_p}}$$

v_n = harmonic flow coefficient, Ψ_n = event plane or flow angle

- $\mathcal{E}_n \equiv (\varepsilon_n, \Phi_n)$, $V_n \equiv (v_n, \Psi_n)$ and $V_n(p_T, y) = (v_n(p_T, y), \Psi_n(p_T, y))$ fluctuate from event to event according to some distributions $P(\mathcal{E}_n)$, $P(V_n)$, $P(V_n(p_T, y))$

- V_n, $V_n(p_T, y)$ cannot be measured in a single event with any kind of precision

  $\rightarrow$ need to measure averages

- $\langle V_n \rangle = 0 = \langle \mathcal{E}_n \rangle$ due to angular average over random orientation of impact parameter $\vec{b} \leftrightarrow b e^{i\Psi_R}$

$\Rightarrow$ only correlations between two or more particles, constructed in such a way that Ψ_R drops out event by event, yield nonzero results

$\Rightarrow$ all flow measurements measure moments of $P(V_n)$; none of them can measure $\langle v_n \rangle$!

3) $\{\varepsilon_n\} \xrightarrow{\text{hydrodynamics}} \{v_n\}$

pressure gradients $\nabla_\mu p(r,\varphi,z) \sim \nabla_\mu e(r,\varphi,z)$

are the hydrodynamic forces that accelerate the hot and dense medium

$\{v_n\}$ = (up to thermal fluctuations, presently not included) deterministic hydrodynamic response to $\{\varepsilon_n\}$

Each event creates its own LITTLE BANG with its own $\{v_n\}$ fingerprint.

4) Show Zhi Qiu hydro movie

5) Relatitivistic fluid dynamics

$\partial_\mu j^\mu_B = 0$ baryon number conservation

$\partial_\mu T^{\mu\nu} = 0$ energy-momentum conservation

Fluid 4-velocity: time-like eigenvector of $T^{\mu\nu}$ (Landau frame):

$T^{\mu\nu} u_\nu = e\, u^\mu$ $\quad u^\mu u_\mu = 1$ $\quad e$ = LRF energy density

LRF = local rest frame; $u^\mu_{LRF} = (1, \vec{0})$

Hydrodynamic decomposition:

$j^\mu_B = n_B u^\mu + V^\mu$ $\quad$ ($u_\mu V^\mu = 0$, purely spatial in LRF)

$$T^{\mu\nu} = e\, u^\mu u^\nu - \underbrace{\left(p(e) + \Pi\right)}_{\mathcal{P}} \Delta^{\mu\nu} + \pi^{\mu\nu}$$

LRF temporal projector

LRF spatial projector

shear stress

bulk viscous pressure

9 dissipative components: $\Pi(1) \oplus \pi^{\mu\nu}(5) \oplus V^\mu(3)$

$$u^\mu u^\nu \xrightarrow{LRF} \begin{pmatrix} 1 & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix} \quad \text{temporal projector}$$

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \xrightarrow{LRF} \begin{pmatrix} 0 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \quad \text{spatial projector}$$

$$\Delta^{\mu\alpha} \Delta_\alpha{}^\nu = \Delta^{\mu\nu}$$

$$\Delta^{\mu\nu} = -X^\mu X^\nu - Y^\mu Y^\nu - Z^\mu Z^\nu \equiv -\Xi^{\mu\nu} - l^\mu l^\nu$$

$$X^\mu \xrightarrow{LRF} (0,1,0,0)$$
$$Y^\mu \xrightarrow{LRF} (0,0,1,0)$$
$$Z^\mu \xrightarrow{LRF} (0,0,0,1)$$

Projections:

$n_B = u_\mu j^\mu_B$

$V^\mu = \Delta^{\mu\nu} j_{\nu,B}$

$e = u_\mu T^{\mu\nu} u_\nu$

$$\Pi = -\frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu} - p(e) \equiv \mathcal{P} - p(e)$$

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta} \equiv T^{\langle\mu\nu\rangle}$$

where $\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\left(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$

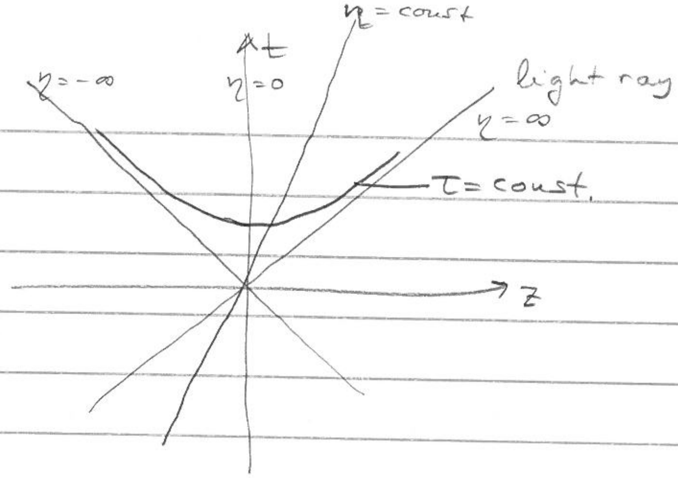
double projector (traceless, spacelike)

$$\Delta^{\mu\nu}_{\alpha\beta}\Delta^{\alpha\beta}_{\gamma\varepsilon} = \Delta^{\mu\nu}_{\gamma\varepsilon}$$

$$j^\mu_B \xrightarrow{LRF} \begin{pmatrix} n_B \\ \vec{V} \end{pmatrix}$$

$$T^{\mu\nu} \xrightarrow{LRF} \underbrace{\begin{pmatrix} e & 0 & 0 & 0 \\ 0 & p(e) & 0 & 0 \\ 0 & 0 & p(e) & 0 \\ 0 & 0 & 0 & p(e) \end{pmatrix}}_{\text{ideal}} + \underbrace{\begin{pmatrix} 0 & & & 0 \\ & \Pi & & \\ & & \Pi & \\ 0 & & & \Pi \end{pmatrix}}_{\text{bulk viscous}} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \boxed{\pi^{ij}} & \\ 0 & & & \end{pmatrix}}_{\text{shear viscous}}$$

6) Milne coordinates



$$\eta \equiv \eta_s = \frac{1}{2}\ln\frac{t+z}{t-z} = \frac{1}{2}\ln\frac{1+z/t}{1-z/t}$$

space-time rapidity

$$\tau = \sqrt{t^2-z^2}$$

longitudinal proper time

(τ, η) span forward light cone of collision "point"

(at infinite energy, nuclei get Lorentz contracted to infinitesimal thickness)

7) Bjorken model:

At very early times $\approx$ boost invariance; transverse homogeneity

under boosts $\quad \eta_s \to \eta_s + y_B \quad , \quad y_{flow} = \frac{1}{2}\ln\frac{1+v_z}{1-v_z} \to y_{flow} + y_B$

$y_{flow} - \eta_s$ is boost invariant; at $z=0$, $\eta_s = y_{flow}$

$$\Rightarrow \boxed{y_{flow} = \eta_s \quad \text{or} \quad v_{z,flow} = z/t}$$

$$\Rightarrow u^\mu = \gamma(1, \vec{v}) \quad \text{with} \quad v_z \approx z/t, \ \vec{v}_\perp \approx 0$$

$(u^\tau, u^x, u^y, u^\eta) = (1, \vec{0})$ in Milne coordinates

("static" flow)

transverse homogeneity, boost invariance $\to V^\mu = 0$, $\pi^{\mu\nu}$ reduces to single nonzero component

$$\pi^{zz} = \tau^2 \pi^{\eta\eta} = -\pi < 0$$

$$\Rightarrow T^{\mu\nu}_{LRF} \underset{\substack{\uparrow \\ \text{early} \\ \text{times}}}{\approx} \begin{pmatrix} e & & & 0 \\ & \mathcal{P}+\pi/2 & & \\ & & \mathcal{P}+\pi/2 & \\ 0 & & & \mathcal{P}-\pi \end{pmatrix} \qquad \mathcal{P} = p(e) + \Pi$$

For massless ("conformal") systems: $T^\mu_{\ \mu} = 0 \Rightarrow \Pi = 0, \ e = 3p(e)$

In the Navier-Stokes limit $\pi^{\mu\nu} = 2\eta\,\sigma^{\mu\nu} = 2\eta\,\underbrace{\nabla^{\langle\mu}u^{\nu\rangle}}_{\text{velocity shear}}$

$\Pi = -\zeta\theta = -\zeta\,\underbrace{\nabla\cdot u}_{\text{scalar expansion rate}}$

$$\pi^{zz}_{NS} = 2\eta\,\sigma^{zz} \underset{\substack{\uparrow\\ \text{Bjorken}}}{=} -\frac{4\eta}{3\tau} = \tau^2\pi^{\eta\eta}$$

$$\Pi_{NS} = -\zeta\theta = -\frac{\zeta}{\tau}$$

8) Hydrodynamic equations $\quad \left(\dot{f} = u^\mu\partial_\mu f = Df\right)$

$$\partial_\mu j^\mu = 0 \quad \Rightarrow \quad \dot{n}_B = -n_B\,\theta - \nabla\cdot V$$

$$\partial_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad \dot{e} = -\left(e + p(e) + \Pi\right)\theta + \pi^{\mu\nu}\nabla_{\langle\mu}u_{\nu\rangle} \quad \text{(time-like)}$$

$$\dot{u}^\mu = \frac{1}{e + p(e) + \Pi}\left(\nabla^\mu(p+\Pi) - \Delta^{\mu\nu}\nabla^\sigma\pi_{\nu\sigma} + \pi^{\mu\nu}\dot{u}_\nu\right)$$

(space-like)

If nothing depends on x, y, η_s (Bjorken symmetry)

$\dot{u}^\mu = 0$ (Bjorken flow solves hydro equations

$$\dot{f} = \frac{df}{d\tau}$$

$$\dot{e} = -\frac{e + p(e) + \Pi - \pi}{\tau}$$

Missing: EOM for π, Π (in the general case: $\pi^{\mu\nu}, \Pi, V^\mu$)

EOM for dissipative flows controlled by microscopic dynamics, not conservation laws

$\boxed{\text{We will derive them from kinetic theory}}$

9) Hydrodynamics from kinetic theory

Boltzmann equation:

$$\left(p_\mu \partial^\mu + m F_\mu \partial_p^\mu\right) f = C[f] \qquad f = f(x,p)$$

F_μ: external or mean field force; $C[f]$: Collision term

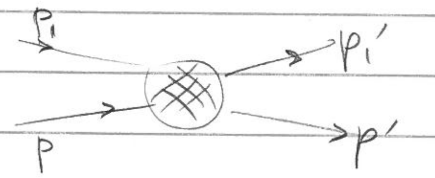
Relaxation time approx: $C[f] = -\dfrac{f(x,p) - f_{eq}(x,p)}{\tau_r(x)}$

Boltzmann collision term

$$C[f] = \frac{1}{2}\int_{p_1 p' p_1'} \delta(p+p_1-p'-p_1')\,\sigma(s,\vartheta)\,\Big[\underbrace{f' f_1' (1\pm f)(1\pm f_1)}_{\text{gain}} - \underbrace{f f_1 (1\pm f')(1\pm f_1')}_{\text{loss}}\Big]$$

valid approach if weakly coupled, nearly on-shell particles

$$s = (p+p_1)^2 = (p'+p_1')^2, \qquad \cos\vartheta = \frac{(p-p_1)\cdot(p'-p_1')}{(p-p_1)^2}$$



(scattering angle in cm system)

$$j^\mu(x) = \int_p p^\mu f(x,p) \equiv \langle p^\mu \rangle \qquad \int_p \equiv \frac{1}{(2\pi)^3}\int \frac{d^3p}{E_p}$$

$$T^{\mu\nu}(x) = \int_p p^\mu p^\nu f(x,p) \equiv \langle p^\mu p^\nu \rangle$$

$$e = u_\mu T^{\mu\nu} u_\nu = \langle (u\cdot p)^2 \rangle \qquad \pi^{\mu\nu} = \langle p^{\langle\mu} p^{\nu\rangle} \rangle$$

$$n = u_\mu j^\mu = \langle (u\cdot p) \rangle \qquad \mathcal{P} = p(e) + \Pi = -\frac{1}{3}\langle \Delta_{\mu\nu} p^\mu p^\nu \rangle$$

$$\mathcal{P}_L = \langle (l\cdot p)^2 \rangle = p + \Pi + \pi^{zz} \qquad = +\frac{\langle \vec{p}_{LRF}^{\,2} \rangle}{3}$$

$$\mathcal{P}_T = \frac{1}{2}\langle p\cdot \Xi \cdot p \rangle$$

$\mathcal{P}_L$, $\mathcal{P}_T$: transverse and longitudinal pressures

Standard viscous hydrodynamics:

$$f(x,p) = f_{eq}(x,p) + \delta f(x,p)$$

$$f_{eq}(x,p) = e^{[\mu(x) - p\cdot u(x)]/T(x)}$$

$$T(x) = ? \quad \mu(x) = ? \quad u^\mu(x) = ?$$

Landau matching: $T^{\mu\nu} u_\nu = e u^\mu \longrightarrow u^\mu$

$$\left.\begin{array}{l} \langle (u\cdot p)^2 \rangle \overset{!}{=} \langle (u\cdot p)^2 \rangle_{eq} \quad \text{or} \quad \langle (u\cdot p)^2 \rangle_{\delta f} = 0 \\ \langle (u\cdot p) \rangle \overset{!}{=} \langle (u\cdot p) \rangle_{eq} \quad \text{or} \quad \langle u\cdot p \rangle_{\delta f} = 0 \end{array}\right\} \mu \,\&\, T$$

$$\Rightarrow \quad V^\mu \equiv \langle p^{\langle\mu\rangle} \rangle_{\delta f} \qquad p^{\langle\mu\rangle} \equiv \Delta^{\mu\nu} p_\nu$$

$$\Pi \equiv -\frac{1}{3} \langle \Delta^{\alpha\beta} p_\alpha p_\beta \rangle_{\delta f} \qquad \left(p(e) = -\frac{1}{3} \langle \Delta^{\alpha\beta} p_\alpha p_\beta \rangle_{eq} \right)$$

$$\pi^{\mu\nu} = \langle p^{\langle\mu} p^{\nu\rangle} \rangle_{\delta f}$$

all dissipative flows arise from δf

Anisotropic viscous hydrodynamics:

$$f(x,p) = f_a(x,p) + \delta\tilde{f}(x,p)$$

$$f_a(x,p) = e^{\left[\tilde{\mu}(x) - \sqrt{(p\cdot u(x))^2 + \xi(x)\,(p\cdot l(x))^2}\right]/\tilde{T}(x)}$$

$$l^\mu(x) \equiv Z^\mu(x) \xrightarrow{LRF} (0,0,0,1)$$

$$\tilde{T}(x) = ? \quad \tilde{\mu}(x) = ? \quad \xi(x) = ? \quad u^\mu(x) = ?$$

u^μ: $\quad T^{\mu\nu} u_\nu = e u^\mu$ $\quad$ time-like eigenvector of $T^{\mu\nu}$

$u^\mu u_\mu = 1$

$\Lambda(x), \tilde{\mu}(x)$: $\quad \langle (u\cdot p)^2 \rangle = \langle (u\cdot p)^2 \rangle_a = \langle (u\cdot p)^2 \rangle_{eq}$

or $\langle (u\cdot p)^2 \rangle_{\delta\tilde{f}} = 0$

$$\langle u\cdot p \rangle = \langle u\cdot p \rangle_a = \langle u\cdot p \rangle_{eq}$$

for massless particles (Martinez + Strickland)

$$\Rightarrow \quad \tilde{T} = T / R_{000}^{1/4}(\xi)$$

$$R_{nlq}(\xi) = \frac{1}{2\,(1+\xi)^{(n-2q)/2}} \int_{-1}^{1} d\cos\alpha \,(\sin\alpha)^{2q} (\cos\alpha)^{l} \times \left[(1+\xi)\sin^2\alpha + \cos^2\alpha\right]^{(n-l-2q-1)/2}$$

$\xi(x)$: $\quad P_L = \langle (l\cdot p)^2 \rangle = \langle (l\cdot p)^2 \rangle_a$ or $\langle (l\cdot p)^2 \rangle_{\delta\tilde{f}} = 0$

Alternatively, can fix $\xi(x)$ by deriving EOM from higher-order moments of the BE

(e.g. Nopoush, Ryblewski, Strickland, PRD 91 ('15) 045007

10) EOM for the (residual) dissipative flows from BE:

$$\dot{\tilde{\Pi}}^{(\sim)} = -\frac{1}{3} D \int_p p\cdot\Delta\cdot p \,\delta \tilde{f}^{(\sim)} = -\frac{1}{3} D \int_p \left(m^2 - \underbrace{(u\cdot p)^2}_{0}\right) \delta \tilde{f}^{(\sim)}$$

0 due to $\langle (u\cdot p)^2\rangle_{\delta \tilde{f}^{(\sim)}} = 0$

$$\dot{\tilde{V}}^{(\sim)\langle\mu\rangle} = \Delta^{\mu}_{\nu} D \int_p p^{\langle\nu\rangle} \delta \tilde{f}^{(\sim)}$$

$$\dot{\tilde{\pi}}^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} D \int_p p^{\langle\alpha} p^{\beta\rangle} \delta \tilde{f}^{(\sim)}$$

with $$\dot{\left(\delta \tilde{f}^{(\sim)}\right)} = -\dot{f}_0 - \frac{1}{u\cdot p}\, p\cdot\nabla\left(f_0 + \delta \tilde{f}^{(\sim)}\right) + \frac{C[f]}{u\cdot p}$$

need to approximate this!

$\Rightarrow$ lot's of work and algebra!

11) Illustration: Gubser flow for a conformal system

(Martinez, McNelis, UH PRC 95 (2017) 054907) MMH

simplification: $\Pi = \tilde{\Pi} = 0$

only one non-zero shear component $\pi = -\tau^2 \pi^{\eta\eta}$

no conserved charge $\longrightarrow \mu = 0$

$(\hat{u}^\rho, \hat{u}^\theta, \hat{u}^\varphi, \hat{u}^\eta) = (1, 0, 0, 0)$ static flow in de Sitter coordinates

all space-time quantities depend only on ρ

all momentum dependence through $\hat{p}_\Omega^2$ and $\hat{p}^\eta$ only
where $\hat{p}_\Omega^2 = \hat{p}_\theta^2 + \hat{p}_\varphi^2/\sin^2\theta$, $\hat{p}_\eta \equiv w = \tau m_\perp \, \mathrm{sh}(y-\eta)$

Denicol, Molnar, Niemi, Rischke, PRD 85 (2012) 114047

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta + \mathcal{J} + \mathcal{K} + \mathcal{R}$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{K}^{\mu\nu} + \mathcal{R}^{\mu\nu}$$

$$\tau_n \dot{V}^{\langle\mu\rangle} + V^\mu = \kappa_n \nabla^\mu\left(\frac{\mu}{T}\right) + \mathcal{J}^\mu + \mathcal{K}^\mu + \mathcal{R}^\mu$$

for viscous hydro, $f = f_{eq} + \delta f$

$\mathcal{J}$: Israel Stewart terms $O(K_n \cdot Re^{-1})$

$\mathcal{K}$: $O(K_n^2)$

$\mathcal{R}$: $O(Re^{-2})$

$$K_n = \frac{\lambda_{micro}}{L_{macro}} = \tau_r \theta \qquad Re^{-1} = \frac{\Pi}{p(e)}, \frac{\sqrt{\pi^{\mu\nu}\pi_{\mu\nu}}}{p(e)}$$

Notation: $\hat{O}$ = observable in de Sitter coordinates, made unitless by appropriate factors of τ

$$\hat{\bar{\pi}} \equiv \frac{\hat{\pi}}{\frac{4}{3}\hat{e}} = \frac{\hat{\pi}}{4\hat{p}(\hat{e})} \qquad \text{normalized shear stress}$$

$$\hat{\pi} \equiv \hat{\pi}^{\eta\eta}$$

Energy-momentum conservation:

$$\partial_\rho \hat{e} + \frac{8}{3}\hat{e}\tanh\rho = \hat{\pi}\tanh\rho \iff \partial_\rho \ln\hat{e} = \frac{4}{3}(\hat{\bar{\pi}}-2)\tanh\rho$$

α) Viscous hydro (DNMR):

$$\partial_\rho \hat{\bar{\pi}} + \frac{\hat{\bar{\pi}}}{\hat{\tau}_r} = \frac{4}{3}\tanh\rho\left(\frac{1}{5} + \frac{5}{14}\hat{\bar{\pi}} - \hat{\bar{\pi}}^2\right)$$

β) aHydro, P_L-matching

$$\partial_\rho \hat{\bar{\pi}} + \frac{\hat{\bar{\pi}}}{\hat{\tau}_r} = \frac{4}{3}\tanh\rho\left(\frac{5}{16} + \hat{\bar{\pi}} - \hat{\bar{\pi}}^2 - \frac{19}{16}\mathcal{F}(\hat{\bar{\pi}})\right)$$

where $\mathcal{F}(\hat{\bar{\pi}}) = \dfrac{\hat{R}_{240}(\xi(\hat{\bar{\pi}}))}{\hat{R}_{200}(\xi(\hat{\bar{\pi}}))}$

to get $\xi(\hat{\bar{\pi}})$, invert $\hat{\bar{\pi}}(\xi) = \dfrac{3I_{220} - I_{200}}{4I_{200}} = \dfrac{1}{4}\left(\dfrac{3\hat{R}_{220}(\xi)}{\hat{R}_{200}(\xi)} - 1\right)$

γ) NRS prescription:

solve $\partial_\rho \xi + \dfrac{\xi(1+\xi)^{3/2}\hat{R}_{200}^{5/4}(\xi)}{\hat{\tau}_r} = -2\tanh\rho\,(1+\xi)$

and then get $\hat{\bar{\pi}}(\xi) = \dfrac{1}{4}\left(\dfrac{3\hat{R}_{220}(\xi)}{\hat{R}_{200}(\xi)} - 1\right)$

δ) NLO-NRS: Use $\xi(\rho)$ from NRS, but add residual contribution to P_L:

$$\hat{P}_L = \langle(\hat{l}\cdot\hat{p})^2\rangle_a + \hat{\tilde{\pi}} = \hat{P}_L^{RS} + \hat{\tilde{\pi}} = \hat{I}_{220}(\hat{\Lambda},\hat{\xi}) + \int_{\hat{p}} \hat{p}_z^2\,\delta\tilde{f}$$

using 14-moment approximation for $\delta\tilde{f}$ (see MMH for details):

$$\partial_\rho \hat{\tilde{\pi}} = -\frac{\hat{\pi}_{RS} + \hat{\tilde{\pi}}}{\hat{\tau}_r} + \dots \quad \text{(lengthy)}$$

Note that adding $\hat{\tilde{\pi}}$ to $\hat{P}_L$ changes the form of the energy conservation law:

$$\underbrace{\partial_\rho \hat{I}_{200}(\hat{\Lambda},\hat{\xi})}_{\hat{e}} + \tanh\rho\,\overbrace{\left(3\hat{I}_{200}(\hat{\Lambda},\hat{\xi}) - \hat{I}_{220}(\hat{\Lambda},\hat{\xi})\right)}^{8/3\,\hat{e} - \hat{\tilde{\pi}}} = \hat{\tilde{\pi}} \tanh\rho$$

ε) Exact solution of the RTA BE:

$$\hat{e}(\rho) = \overbrace{\vphantom{x}}^{D(\rho,\rho_0)\left(\frac{\cosh\rho_0}{\cosh\rho}\right)^4}\hat{e}_{RS}(\hat{\Lambda}_0, \xi_{FS}(\rho;\rho_0,\xi_0)) + \frac{1}{c}\int_{\rho_0}^{\rho} d\rho'\, D(\rho,\rho')\, T(\rho') \left(\frac{\cosh\rho'}{\cosh\rho}\right)^4 \hat{e}_{RS}(\hat{T}(\rho'); \xi_{FS}(\rho;\rho',0))$$

$$\hat{\pi}(\rho) = \dots \hat{\pi}_{RS}(\dots) + \frac{1}{c}\int d\rho' \dots \hat{\pi}_{RS}(\hat{T}(\rho'); \xi_{FS}(\rho;\rho',0))$$

$$\xi_{FS}(\rho;\rho_a,\xi_a) = -1 + (1+\xi_a)\left(\frac{\cosh\rho_a}{\cosh\rho}\right)^2$$