The Equation of State of Dense Matter and Neutron Star Masses and Radii

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Use down and up arrows to proceed to the next or previous slide. Problems are at the end. A pdf version is available. Underlined references contain links to relevant papers or online resources. Please feel free to email me with questions or corrections.

Outline

Goal: Pursue problem with some depth while still introducting generic tools

- Neutron stars
- Thermodynamics and statistical mechanics
- Density functionals and Skyrme
- Infinite nucleonic matter and nuclei
- Weak equilibrium
- Newtonian and GR stars
- χ^2 fitting and Bayesian inference

The QCD phase diagram

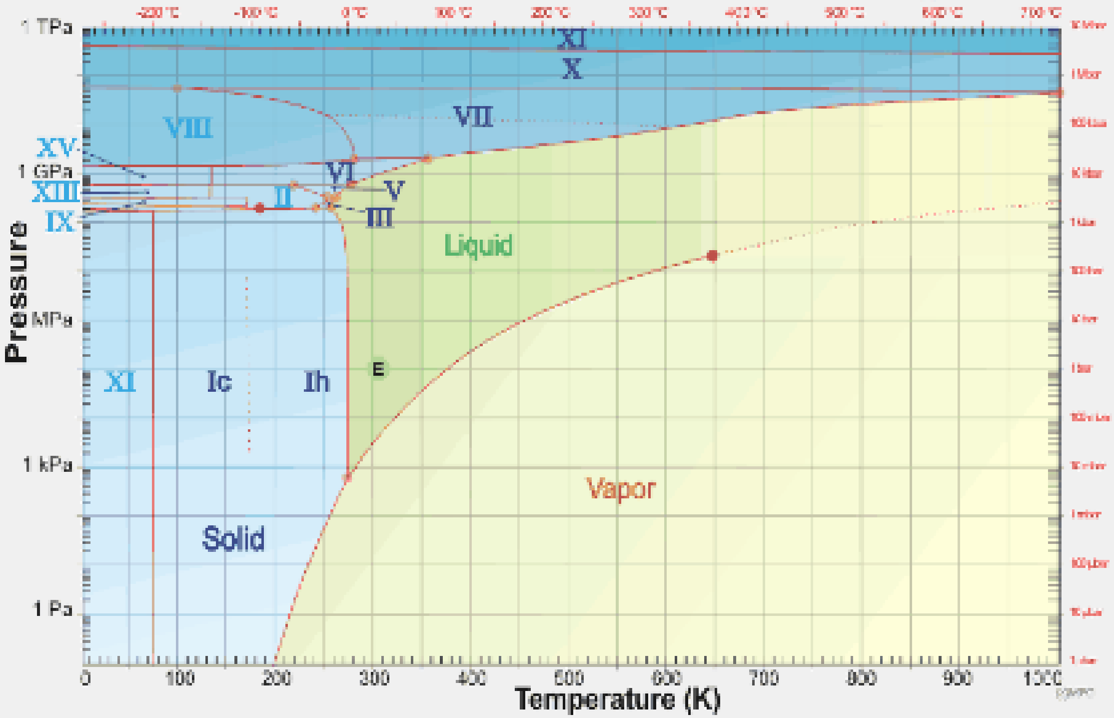
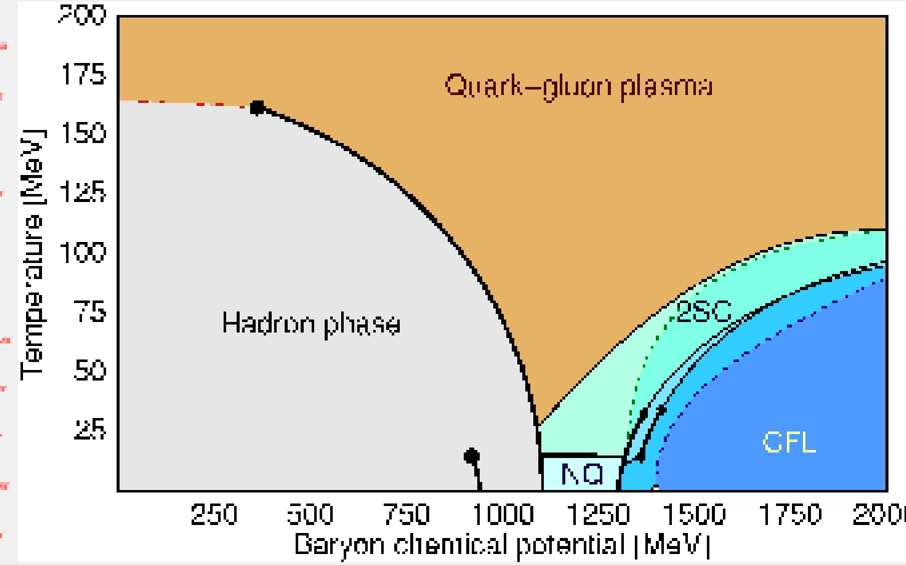


Figure from Martin Chaplin

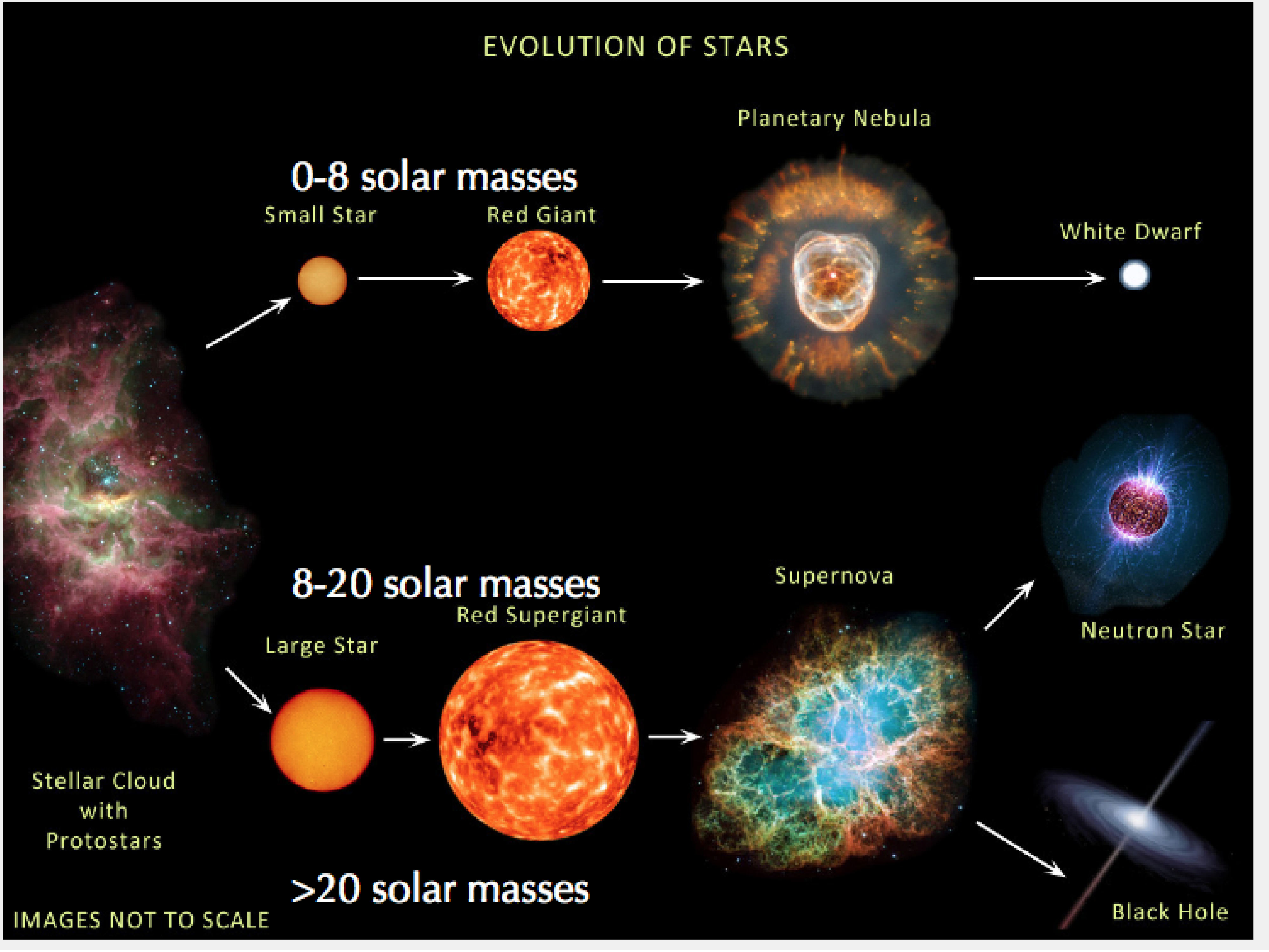


Rüster et al. (2006)

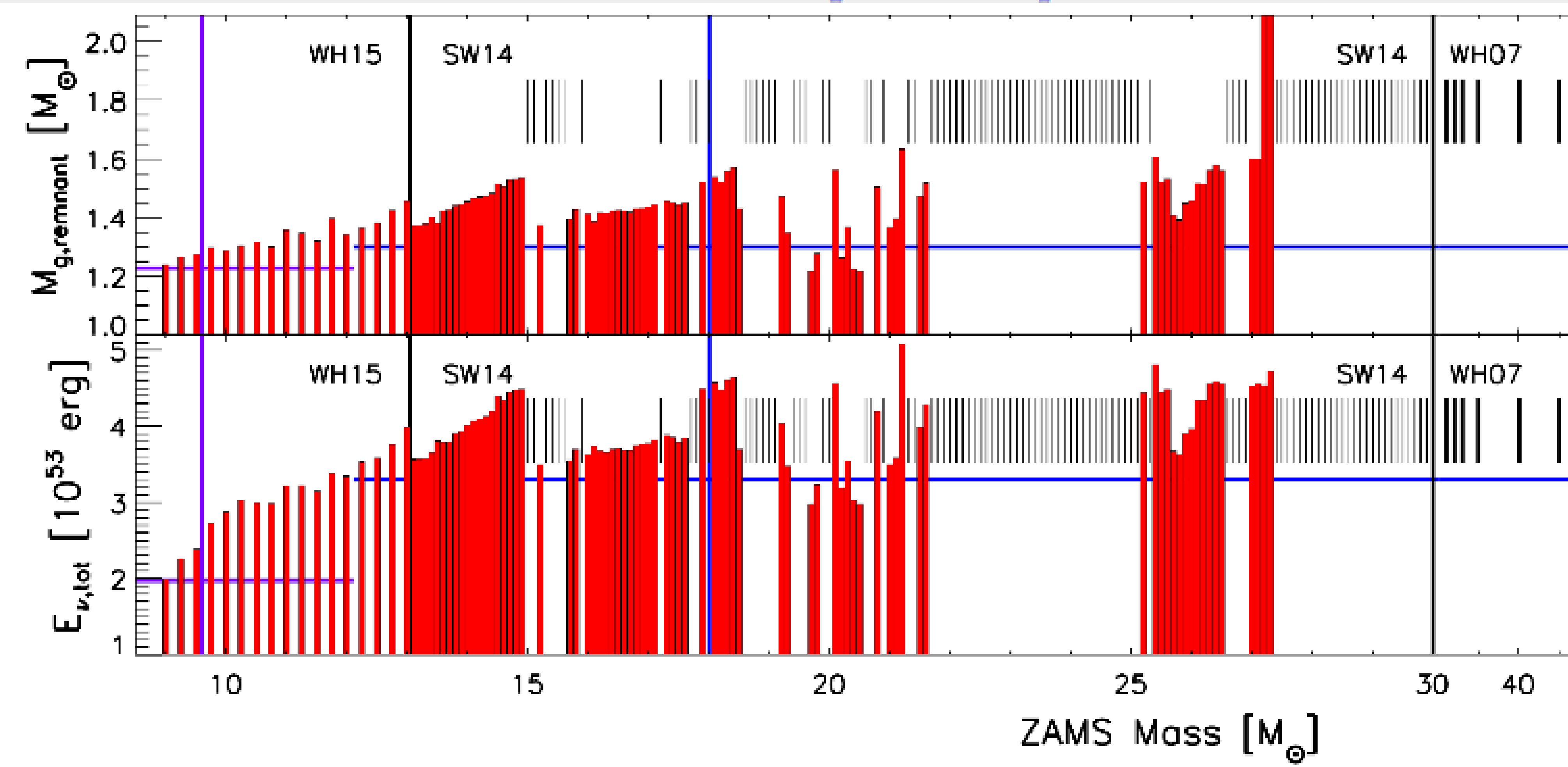
- Heavy-ion collisions and lattice QCD sensitive primarily to high T, low μ regions

- Electromagnetic and gravitational wave observations of neutron star-related phenomena are the **best** probe of cold, dense (and non-perturbative) QCD.

Stellar Evolution



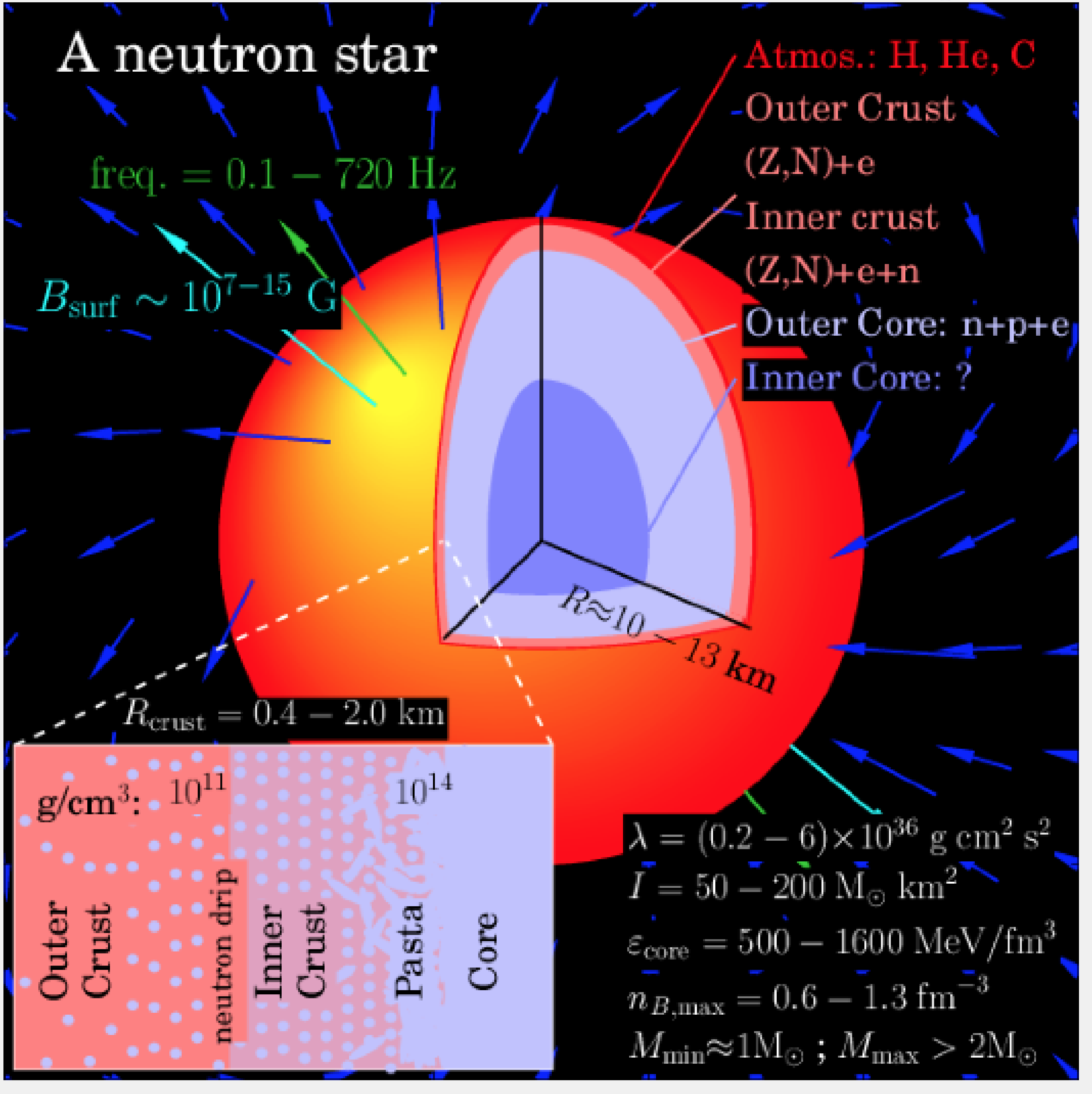
Fate of Core-collapse Supernovae



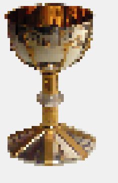
Sukhbold et al. (2016)

- Gravitational mass of the remnant and the total energy released
- Still a lot of uncertainty

Neutron Star Composition



What is the composition of the neutron star core?



Inspired by D. Page; open source (python)

Infinite nucleonic matter

- Think of a large number of neutrons and protons in a box. What is the energy per particle for that box?
- Electrons and muons: always present to ensure charge neutrality
- Presume local thermodynamic equilibrium
- Can ignore gravity in the computation of microscopic properties of matter: gravitational potential change is small over small scales

Thermodynamic preliminaries

$$E = -PV + TS + \sum_i \mu_i N_i \quad ; \quad dE = -PdV + TdS + \sum_i \mu_i dN_i$$

- Natural variables for internal energy, E, are S, V, and N: internal energy is minimized at fixed S, V, and N.

- Helmholtz free energy

$$F = E - TS \qquad dF = dE - TdS - SdT$$

$$\Rightarrow dF = -SdT - PdV + \sum_i \mu_i dN_i$$

- Free energy is minimized at fixed N, V and T

- Neutron star case, $T = 0$ and $\sum_i \mu_i n_i = \mu_B n_B$:

$$P = n_B^2 \frac{\partial(E/n_B)}{\partial n_B}$$

Quantum Statistical Mechanics

- Use units where $\hbar = c = 1$; Start with non-interacting particles

$$P(\mu, T) = \pm g T \int \frac{d^3k}{(2\pi)^3} \log\left[1 \pm e^{-(E-\mu)/T}\right] \quad E = \sqrt{k^2 + m^2}$$

Johns, Ellis, and Lattimer (1996)

- Upper signs for fermions; constant g is spin degeneracy factor
- For fermions: degenerate limit $\mu \to \infty$; non-degenerate limit $\mu \to -\infty$, unless antiparticles are included, then $\mu \to 0$.
- For relativistic systems

$$E - \mu = \sqrt{k^2 + m^2} - \mu$$

- For non-relativistic systems

$$E - \mu = m + \frac{k^2}{2m} - (\tilde{\mu} + m) = \frac{k^2}{2m} - \tilde{\mu}$$

and also the integrals are much easier

Nondegenerate expansion

(this is mostly for reference)

- Define:

$$t \equiv T/m \quad ; \quad \psi \equiv (\mu - m)/T$$

- Using the identity

$$\int_0^\infty \frac{x^4\left(x^2+z^2\right)^{-1/2}\,dx}{1+e^{\sqrt{x^2+z^2}-\phi}} = 3z^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} e^{n\phi} K_2(nz)$$

Tooper (1969)

- one obtains

$$P = \frac{gm^4}{2\pi^2}\left[t^2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} e^{k(\psi+1/t)} K_2\left(\frac{k}{t}\right)\right]$$

- This can be used directly, unless t is large compared to k, in which case one can use

$$\sqrt{\frac{2x}{\pi}}\,e^x K_2(x) \approx 1 + \frac{3}{8x} - \frac{15}{128x^2} + \ldots$$

Degenerate expansion

- (this is mostly for reference)
- Use the Sommerfeld expansion:

$$\int_0^\infty dz\, \frac{f(z)}{1+e^{(z-x)/t}} = \int_0^x f(z) + \sum_{n=1}^{\infty} \pi^{2n} t^{2n} \left[f^{(2n-1)}(x)\right] \left[\frac{2(-1)^{1+n}(2^{2n-1}-1)B_{2n}}{(2n)!}\right]$$

- where B_{2n} are the Bernoulli numbers and $f^{(2n-1)}$ represents the $(2n-1)$th derivative of f .
- This is an asymptotic, not convergent, expansion
- Applying this to the pressure leads to the function ($x \equiv \psi t$)

$$P_0 \equiv \frac{1}{24}(1+x)\sqrt{x(2+x)}\,[-3+2x(2+x)] + \frac{1}{4}\log\left[\frac{\sqrt{x}+\sqrt{2+x}}{\sqrt{2}}\right]$$

- this runs into numerical issues when x is small, but can be replaced by a Taylor series

Second Derivatives of the Pressure

- Three independent derivatives are enough to compute all second derivatives
- In terms of μ and T

$$\left(\frac{\partial n}{\partial \mu}\right)_{V,T}, \quad \left(\frac{\partial s}{\partial T}\right)_{V,\mu}, \quad \text{and} \quad \left(\frac{\partial n}{\partial T}\right)_{V,\mu} = \left(\frac{\partial s}{\partial \mu}\right)_{V,T}$$

- E.g. specific heats

$$C_V = \frac{T}{n}\left[\left(\frac{\partial s}{\partial T}\right)_{\mu,V} - \left(\frac{\partial n}{\partial T}\right)_{\mu,V}^2 \left(\frac{\partial n}{\partial \mu}\right)_{T,V}^{-1}\right]$$

$$c_P = \frac{T}{n}\left(\frac{\partial s}{\partial T}\right)_{\mu,V} + \frac{s^2 T}{n^3}\left(\frac{\partial n}{\partial \mu}\right)_{T,V} - \frac{2sT}{n^2}\left(\frac{\partial n}{\partial T}\right)_{\mu,V},$$

Derivation here

Non-relativistic Energy Density Functionals

- Density functional theory: the ground state of a many-body system uniquely determined by the densities.
- Separate into kinetic and potential energy (ambiguous)

$$\mathcal{H} = \mathcal{H}_{\rm kin}(n_n, n_p) + \mathcal{H}_{\rm pot}(n_n, n_p)$$

- If we can determine the kinetic energy from the non-interacting case:

$$\mathcal{H}_{\rm kin,i} = \frac{g}{2\pi^2}\int_0^{k_{Fi}} k^2\, dk \frac{k^2}{2m_i} = \frac{g k_{Fi}^5}{20\pi^2 m_i} \quad \text{where} \quad n_i = \frac{g}{2\pi^2}\int_0^{k_{Fi}} k^2\, dk$$

- If interactions modify the kinetic energy, then rewrite them as

$$\mathcal{H}_{\rm kin} = \frac{g k_F^5}{20\pi^2 m^*}$$

- Where m^* is an "effective mass" (which may depend on the densities)
- Finite temperature, carry over all of the same temperature integrals, replacing m^* with m (Fermi-Liquid theory) and replacing μ with an effective chemical potential

The Skyrme Interaction

Skyrme (1959), Negele and Vautherin (1972), Stone and Reinhard (2007), Kortelainen et al. (2014)

$$\mathcal{H} = \frac{\tau_n}{2m_n} + \frac{\tau_p}{2m_p} + C_k n_B(\tau_n + \tau_p) + D_k \left(\tau_n n_n + \tau_p n_p\right) + C_{p2} n_B^2 + D_{p2}(n_n^2 + n_p^2) + C_{p3} n_B^{2+\alpha} + \ldots$$

- For non-homogeneous systems, add gradient terms

$$\mathcal{H}_{\mathrm{grad}} = \frac{1}{2}\left[Q_{nn}\left(\vec{\nabla} n_n\right)^2 + 2Q_{np}\vec{\nabla} n_n \cdot \vec{\nabla} n_p + Q_{pp}\left(\vec{\nabla} n_p\right)^2\right]$$

- Can think of a gradient expansion, but they don't always converge
- add also Coulomb, spin-orbit, ...

EOS Near Saturation

- Define $n_B = n_n + n_p$, $x = n_p/n_B$, $\delta = 1 - 2x$, and $\epsilon = (n - n_0)/(3n_0)$

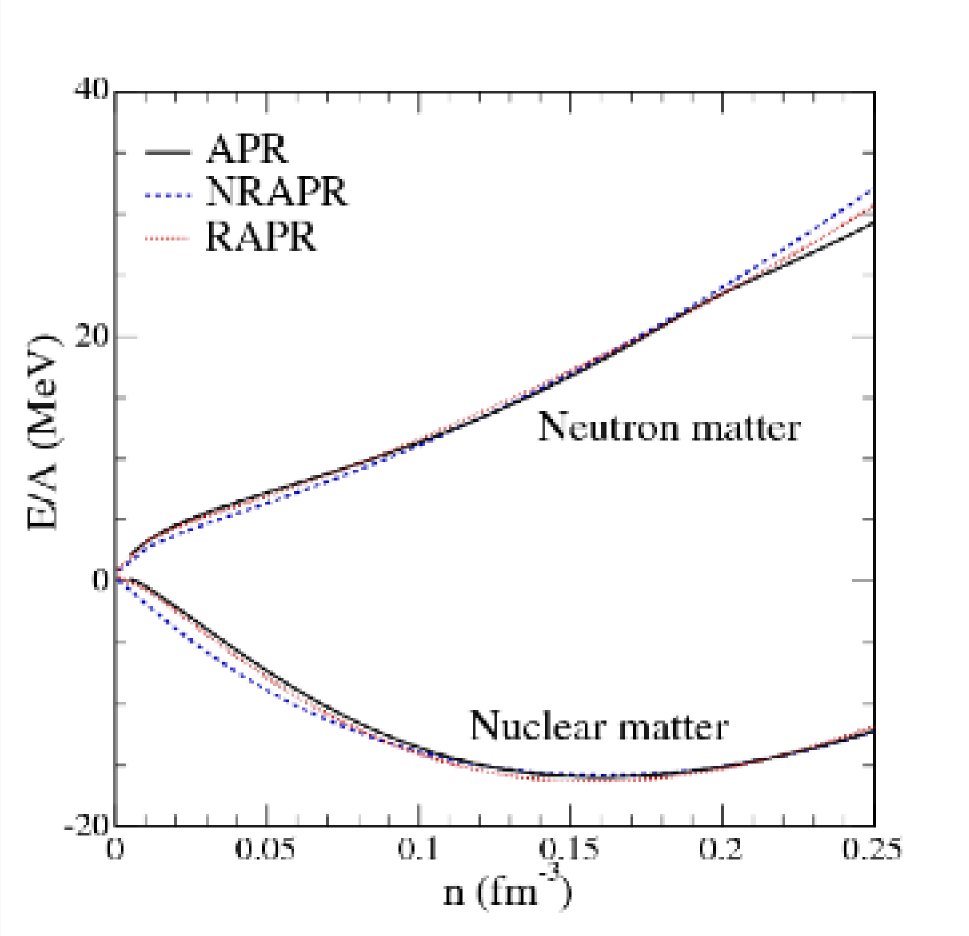
$$E(n_B, \delta) = -B + \frac{K}{2!}\epsilon^2 + \frac{Q}{3!}\epsilon^3 + \delta^2\left(S + L\epsilon + \frac{K_{\rm sym}}{2!}\epsilon^2 + \frac{Q_{\rm sym}}{3!}\epsilon^3\right) + E_4(n_B, \delta) + \mathcal{O}(\delta^6)$$

- where $n_0 \approx 0.16\ \mathrm{fm}^{-3}$ and $B \sim 16$ MeV
- Compression modulus: $\chi = -1/V(dV/dP) = 1/n(dP/dn)^{-1}$
- Incompressibility, $K = 9/(n\chi)$, i.e.

$$K = 9\left(\frac{\partial P}{\partial n_B}\right)_{n_B = n_0}$$

- Incompressibility is measured in giant monopole resonances, $K = 220 - 260$ MeV.

The Nuclear Symmetry Energy



Steiner et al. (2005)

- Define nuclear matter as a box with equal numbers of neutrons and protons
- No protons ⇒ pure neutron matter

- Define the "symmetry energy" as the difference
- $S(n_B) \equiv E_{\mathrm{neut}}(n_B) - E_{\mathrm{nuc}}(n_B)$
- S is the value at the nuclear saturation density $S = S(n_0) = 29$ to 36 MeV
- L is the derivative, $L = 3 n_0 S'(n_0) = 30$ to 70 MeV

Weisacker-Bethe semi-empirical mass formula

$$E(Z,N) = -BA + E_{\rm surf}A^{2/3} + CZ^2A^{-1/3} + S\frac{(N-Z)^2}{A}$$

$$+E_{\rm pair}\begin{cases} +1 & \text{N and Z odd} \\ -1 & \text{N and Z even} \\ 0 & \text{otherwise} \end{cases}$$

von Weisäcker (1935); Bethe and Bacher (1936); Dieperink et al. (2009) Moller et al. (2016)

- Radius $\sim A^{1/3}$ - this is saturation!
- Surface energy $\sim R^2 \sim A^{2/3}$; curvature energy $\sim R \sim A^{1/3}$
- Expansion in $1/R$
- Coulomb length scale = Debye screening length
- Can add "shell effects" via Strutinsky method

Weak Equilibrium

- Over long time scales, weak equilibrium is achieved through $n \leftrightarrow p + e$
- This implies detailed balance, i.e. $\mu_n = \mu_p + \mu_e$
- Weak equilibrium choses a particular n/p ratio
- If we presume that baryon number is conserved, then baryon and charge conservation imply

$$\mu_i = B_i \mu_B + Q_i \mu_Q$$

- where B_i is the baryon number of particle i and Q_i is its charge.
- Neutrinos leave the star unless $T > 10$ MeV, thus no chemical potential

Classical stars

- Begin with $T = \vec{B} = \Omega = 0$ and spherical symmetry

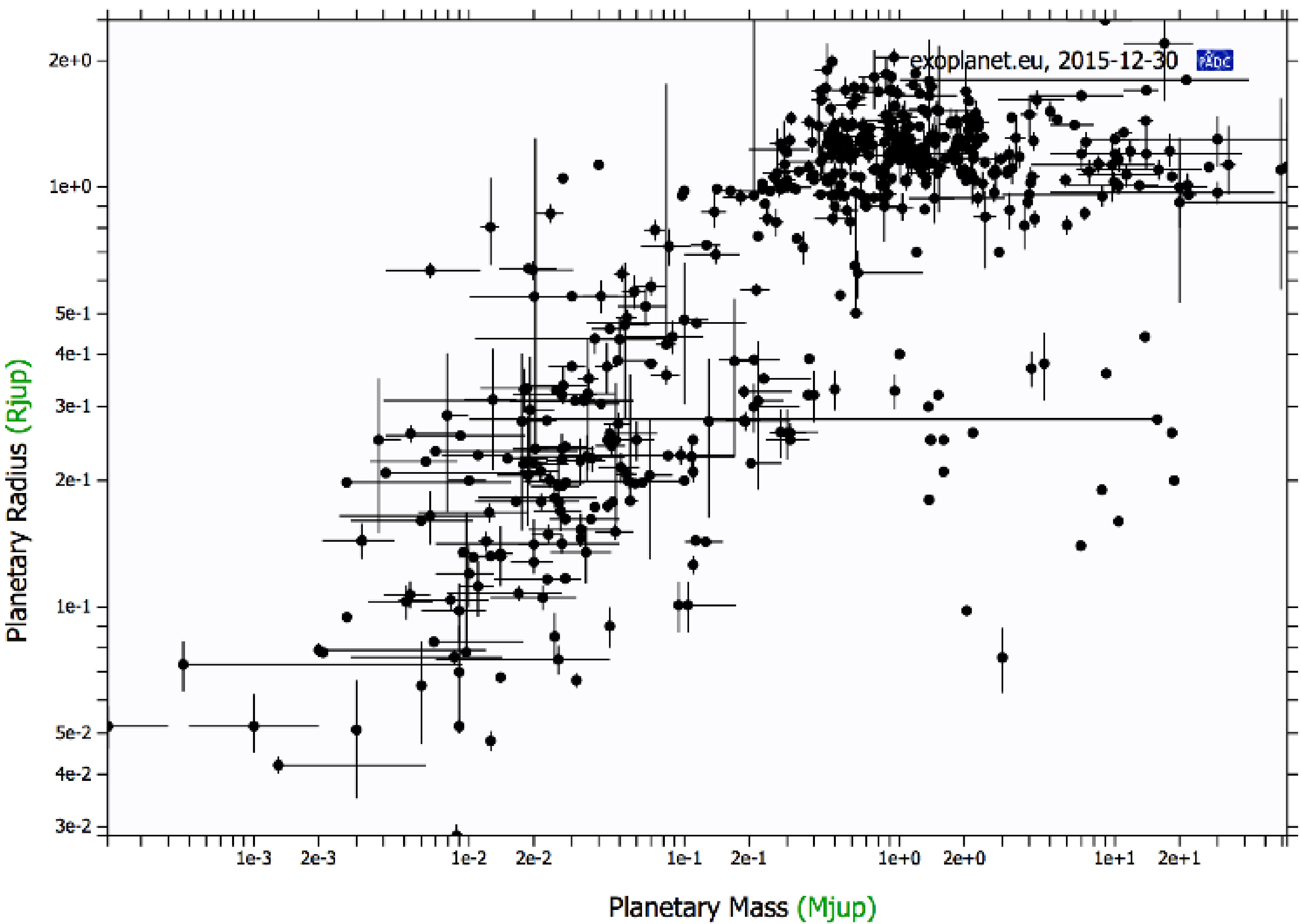
$$\frac{dm}{dr} = 4\pi r^2 \rho; \quad m(r = 0) = 0$$

$$\frac{dP}{dr} = \frac{-Gm\rho}{r^2}; \quad P(r = R) = 0$$

$$M = \int_0^R 4\pi r^2 \rho \, dr$$

- where ρ is the rest mass density
- Stellar structure just an application Newton's laws
- One parameter family of solutions, as long as $P(\rho)$ is specified, parameterized by $P(r = 0)$

Planetary masses and radii



Relativistic stars

- Specify the metric

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

- Now, m is "gravitational mass"

$$\frac{dm}{dr} = 4\pi r^2 \varepsilon; \quad m(r=0) = 0$$

$$\frac{dP}{dr} = \frac{-Gm\varepsilon}{r^2}\left(1 + \frac{P}{\varepsilon}\right)\left(1 + \frac{4\pi P r^3}{m}\right)\left(1 - \frac{2Gm}{r}\right); \quad P(r=R) = 0$$

- The baryonic mass is

$$M_B = \int_0^R 4\pi r^2 n_B m_B \left(1 - \frac{2Gm}{r}\right)^{-1/2} dr$$

- Gravitational potential:

$$\text{outside}: e^{2\Phi} = \left(1 - \frac{2GM}{r}\right) \quad \text{inside}: \frac{d\Phi}{dr} = -\frac{1}{\varepsilon}\frac{dP}{dr}\left(1 + \frac{P}{\varepsilon}\right)^{-1}$$

Problem 1

- Using the definition of the gravitational potential in a zero-temperature relativistic star, and $n = dP/d\mu$, show that if we redefine a new chemical potential which is modified by the GR, this new chemical potential is a constant through the entire star.

Problem 2

- Presume that energy per particle of nucleonic matter is

$$E/A(n_n, n_p) = -B + \frac{K}{9}\left(\frac{n_B - n_0}{n_0}\right)^2 + (1-2x)^2 S(n_B)$$

- with $n_B \equiv n_n + n_p$ and $x = n_p/n_B$.
- Assuming entropy $= T = 0$, obtain the electron chemical potential in beta-equilibrium in terms the constants and functions given above

Problem 3

- The speed of sound, c_s^2 can be obtained from the derivative $c_s^2 = dP/d\varepsilon$.

- Assume a quark matter equation of state of

$$P = \frac{3}{4\pi^2}\mu_q^4 - \frac{3a_2}{4\pi^2}\mu_q^2 + B$$

- where $\mu_q \equiv \mu_B/3$ and $a_2 = m_s^2 - 4\Delta^2$ and m_s is the strange quark mass and Δ is the quark superconducting gap and B is the bag constant. Determine the speed of sound at large μ and determine how it depends on m_s, B and Δ.