Lecture 2 – Neutrino oscillations

1) ν oscill. in vacuum (2ν case)

2) ν oscill. in matter – MSW effect (2ν case)

3) Oscillations of 3ν system

Caveats:

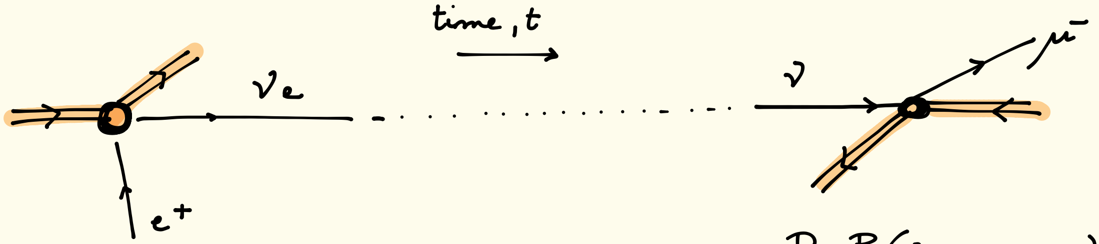
- idealized case of 2ν states $\rightarrow$ can generalize to 3ν (realistic)

- standard, traditional formalism $\rightarrow$ more advanced methods address shortcomings (wavepacket formalism)

1) ν oscill. in vacuum

Periodic change of (measured) flavor, due to quantum propagation

Produced ν_e : detected ν_μ :



$$P = P(\nu_e \to \nu_\mu, t)$$

flavor conversion probability

Toy model: 2 ν system

$\{|\nu_1\rangle, |\nu_2\rangle\}$ mass eigenstates $\qquad \langle \nu_i | \nu_j \rangle = \delta_{ij} \quad (i,j = 1,2)$

$\{m_1, m_2\}$ masses

$\{|\nu_e\rangle, |\nu_\mu\rangle\}$ flavor eigenstates, $\langle \nu_\alpha | \nu_\beta \rangle = \delta_{\alpha\beta} \quad (\alpha, \beta = e, \mu)$

Notation: $|\nu\rangle$ generic state, $\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} \equiv \begin{pmatrix} \langle \nu_e | \nu \rangle \\ \langle \nu_\mu | \nu \rangle \end{pmatrix}$

Mixing matrix:

$$\begin{cases} |\nu_e\rangle = c\,|\nu_1\rangle + s\,|\nu_2\rangle \\ |\nu_\mu\rangle = -s\,|\nu_1\rangle + c\,|\nu_2\rangle \end{cases} \longrightarrow U = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

<u>real</u> (no physical phases for 2 ν)

$c \equiv \cos\vartheta \qquad s \equiv \sin\vartheta$

Hamiltonian in vacuum, in $\{|\nu_i\rangle\}$ basis $(i=1,2)$

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \quad , \text{ time-independent}$$

$$E_i = \sqrt{p^2 + m_i^2} \qquad (\text{same } p \text{ assumption})$$

Quantum evolution : $t=0 \quad |\nu(0)\rangle = |\nu_e\rangle$

Calculate $P(\nu_e \to \nu_\mu, t)$:

$$|\nu(t)\rangle = c\, e^{-iE_1 t} |\nu_1\rangle + s\, e^{-iE_2 t} |\nu_2\rangle$$

$$P(\nu_e \to \nu_\mu, t) = \Big|\big(\underbrace{-s\langle\nu_1| + c\langle\nu_2|}_{\langle\nu_\mu|}\big)|\nu(t)\rangle\Big|^2 = \left|-sc\, e^{-iE_1 t} + sc\, e^{-iE_2 t}\right|^2 =$$

⋮ (steps)

$$P(\nu_e \to \nu_\mu, t) = \sin^2 2\vartheta \, \sin^2\left(\frac{E_2 - E_1}{2} t \right)$$ ← depends on difference of eigenvalues.

Approximate, relativistic limit :

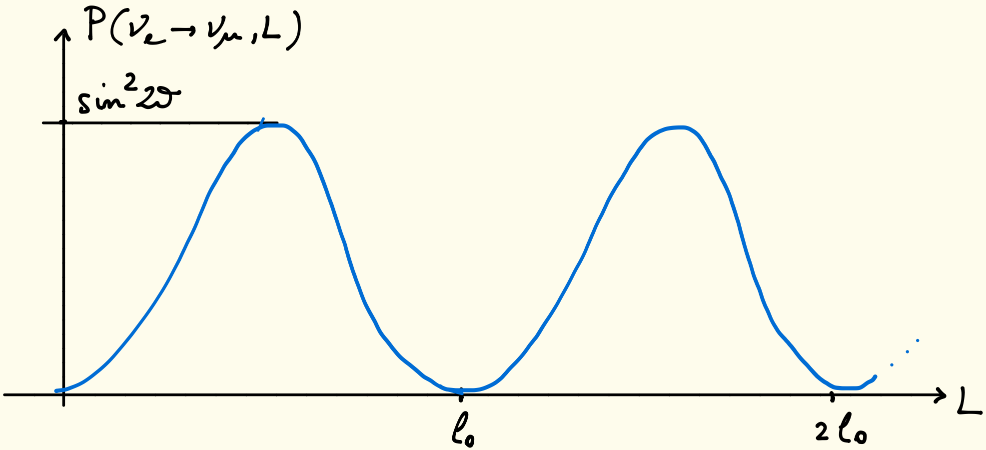
$$E_i = \sqrt{p^2 + m_i^2} \simeq p + \frac{m_i^2}{2p} \simeq p + \frac{m_i^2}{2E} \qquad (p^2 \gg m_i^2)$$

$t \simeq L$ (propagation distance)

↓

$$\boxed{P(\nu_e \to \nu_\mu, L) = \sin^2 2\vartheta \, \sin^2\left(\frac{\pi L}{\ell_0} \right)} \qquad \Delta m^2 \equiv m_2^2 - m_1^2$$

$$\boxed{\ell_0 = \frac{4\pi E}{\Delta m^2}}$$ oscill. length



$$\ell_0 = \frac{4\pi E}{\Delta m^2} \simeq 2480 \text{ km} \left(\frac{E}{\text{GeV}}\right)\left(\frac{10^{-3} \text{ eV}^2}{\Delta m^2}\right)$$

Note:

- $P(\nu_e \to \nu_\mu, L=0) = 0$ as should.

- Non trivial solution requires $\theta \neq 0$ AND $\Delta m^2 \neq 0$

- depends on $\Delta m^2 = m_2^2 - m_1^2$, NOT on absolute mass scale

for 2×2 case only

$$P(\nu_\mu \to \nu_e, L) = P(\nu_e \to \nu_\mu, L)$$

$$P(\nu_e \to \nu_e) = P(\nu_\mu \to \nu_\mu, L) = 1 - P(\nu_e \to \nu_\mu, L)$$

$$P(\bar{\nu}_e \to \bar{\nu}_\mu, L) = P(\nu_e \to \nu_\mu, L)$$ CP-conserving

Averaged oscillations:

consider L fixed, and $P(\nu_e \to \nu_\mu, E)$ (function of energy):

separation between maxima:

$$\pi = \frac{L\Delta m^2}{4E} - \frac{L\Delta m^2}{4(E+\delta E)} \longrightarrow \delta E \simeq \frac{4\pi E^2}{\Delta m^2 L}$$

if $\delta E \ll \varepsilon$ (detector resolution) $\longrightarrow$ oscillations can't be resolved!

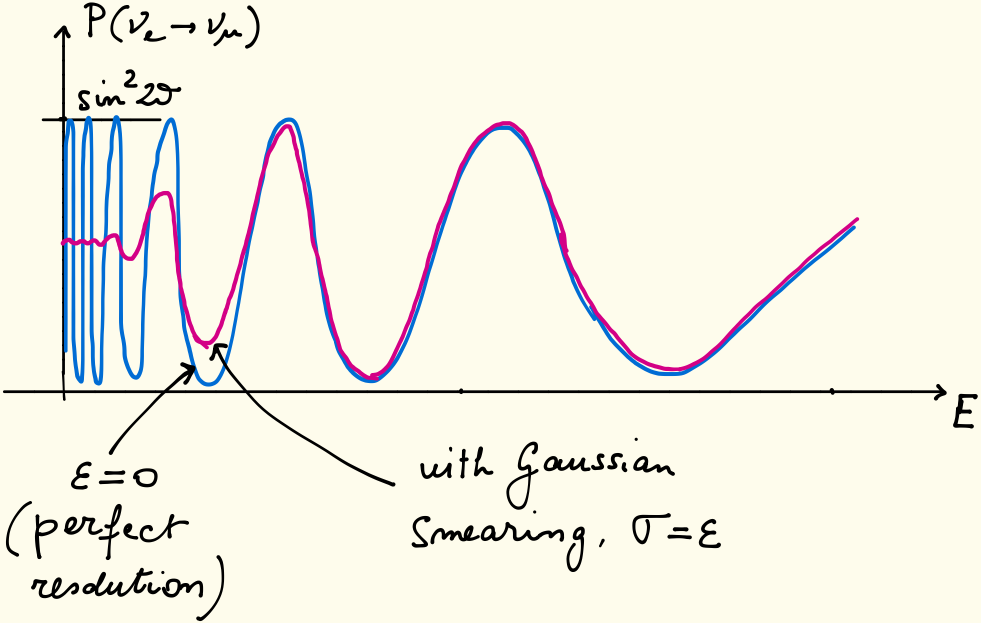
$$\Rightarrow \boxed{P(\nu_e \to \nu_\mu) \simeq \frac{1}{2}\sin^2 2\vartheta} \leq 1/2$$

$\longleftarrow$ <u>avg. vacuum oscill.</u>

<u>can't exceed 50% !</u>

$$\delta E \simeq \frac{4\pi E^2}{\Delta m^2 L} \xrightarrow[L \to \infty]{E \to 0} 0$$

For L fixed:



2) ν oscill. in matter, 2×2 case

$\begin{cases} \text{Coherent scattering (refraction)} : \vec{p}_\nu \text{ unchanged} \\ \text{Incoherent scattering (absorption)} \end{cases}$

$V \propto G_F$

$\sigma \propto G_F^2 E$

(negligible in most applications)

Focus on refraction:

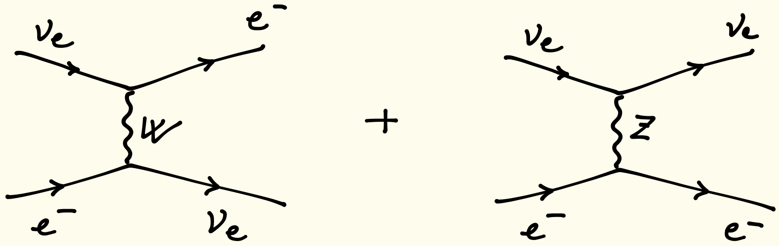
effective treatment : ν in external potential

$$H = H_{vac} + V$$

MSW : Mikheev, Smirnov (1985), Wolfenstein (1978)

2.1 The refraction potential

Example: $\nu_e - e^-$ scattering, n_e = number density of e^-.



Charged Current Neutral Current

C.C. Only: Hamiltonian (after Fierz transformation):

$$\mathcal{H}^{CC}(x) = \frac{G_F}{\sqrt{2}} \left[\bar{\nu}_e(x) \gamma^\rho (1-\gamma^5) \nu_e(x) \right] \left[\bar{e}(x) \gamma_\rho (1-\gamma^5) e(x) \right]$$

Average over electron background (unpolarized):

$$\overline{\mathcal{H}^{cc}} = \frac{G_F}{\sqrt{2}} \left[\bar{\nu}_e(x)\gamma^\rho(1-\gamma^5)\nu_e(x)\right] \int d^3p_e\, f(p_e, T)$$

$$\times \frac{1}{2} \sum_{h_e=\pm1} \langle e^-(\underbrace{p_e, h_e}) | \bar{e}(x)\gamma_\rho(1-\gamma^5)e(x) | e^-(\underbrace{p_e, h_e}) \rangle$$

same!

$f(p_e, T) \leftarrow e^-$ statistical distribution

$$\overline{\mathcal{H}^{cc}}(x) = \underbrace{\sqrt{2}G_F n_e}_{\equiv V_{cc}} \bar{\nu}_{eL}(x)\gamma^0 \nu_{eL}(x)$$

$\hookrightarrow$ potential energy: $\boxed{V_e^{cc} = V_{cc} = \sqrt{2}\, G_F n_e}$

Similarly, for $\bar{\nu}_e - e^-$ scattering:

$$\boxed{V_{\bar{e}}^{cc} = - V_{cc}}$$

Generalize to realistic matter : e^-, p, n

$$n_p = n_e \quad , \quad n_n$$

$$\begin{cases} V_e = V_e^{CC} + V_e^{NC} = \sqrt{2}\, G_F\, n_e - \frac{1}{2}\sqrt{2}\, G_F\, n_n \\ V_\mu = \qquad V_\mu^{NC} \qquad = \qquad -\frac{1}{2}\sqrt{2}\, G_F\, n_n \end{cases}$$

$$\begin{cases} V_{\bar{e}} = V_{\bar{e}}^{CC} + V_{\bar{e}}^{NC} = -\sqrt{2}\, G_F\, n_e + \frac{1}{2}\sqrt{2}\, G_F\, n_n \\ V_{\bar{\mu}} = \qquad V_{\bar{\mu}}^{NC} \qquad = \qquad +\frac{1}{2}\sqrt{2}\, G_F\, n_n \end{cases}$$

Note: contributions of NC on e^- and NC on p cancel.

2.2 Constant density: $V_\alpha(\cancel{x}) = \text{const.} \quad (\alpha = e, \mu)$

H in flavor basis: time-independent

$$H_{fl} = U H_{vac} U^\dagger + V = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix} + \begin{pmatrix} V_e & 0 \\ 0 & V_\mu \end{pmatrix}$$

Solution only depends on difference of eigenvalues $\Rightarrow$ can subtract identity terms.

Take $H_{fl} \longrightarrow H_{fl} - \mathbb{I}\left(p + \frac{m_1^2 + m_2^2}{4E} + \frac{V_e + V_\mu}{2} \right)$

$$\blacktriangleright \; H_{fl} = \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \begin{pmatrix} V_{cc}/2 & 0 \\ 0 & -\frac{V_{cc}}{2} \end{pmatrix}$$

$$V_{cc} = V_e - V_\mu = \sqrt{2}\, G_F n_e$$

$$\boxed{H_{f\ell} = \frac{\Delta m^2}{4E}\begin{pmatrix} -\cos 2\vartheta + x & \sin 2\vartheta \\ \sin 2\vartheta & \cos 2\vartheta - x \end{pmatrix}}$$

$$\boxed{x \equiv \frac{2\sqrt{2}\, G_F\, n_e\, E}{\Delta m^2}}$$

Calculate $P(\nu_\alpha \to \nu_\beta, L)$: use t-independent solution (H_{fl} is t-indep)

Diagonalize H : $\{\nu_{1m}, \nu_{2m}\}$ eigenstates, $\{E_{1m}, E_{2m}\}$ eigenvalues

$$U_m = \begin{pmatrix} c_m & s_m \\ -s_m & c_m \end{pmatrix} \quad \text{mixing matrix}$$

$$\begin{cases} |\nu_e\rangle = c_m |\nu_{1m}\rangle + s_m |\nu_{2m}\rangle \\ |\nu_\mu\rangle = -s_m |\nu_{1m}\rangle + c_m |\nu_{2m}\rangle \end{cases}$$

$c_m \equiv \cos\vartheta_m$, $s_m = \sin\vartheta_m$

$$\boxed{\begin{cases} \sin 2\vartheta_m = \dfrac{\sin 2\vartheta}{\left[\sin^2 2\vartheta + (\cos 2\vartheta - x)^2\right]^{1/2}} \\[2ex] \cos 2\vartheta_m = \dfrac{\cos 2\vartheta - x}{\left[\sin^2 2\vartheta + (\cos 2\vartheta - x)^2\right]^{1/2}} \end{cases}}$$

$$\boxed{E_{2m} - E_{1m} = \frac{\Delta m^2}{2E}\left[\sin^2 2\vartheta + (\cos 2\vartheta - x)^2\right]^{1/2}}$$

$$x \equiv \frac{2\sqrt{2}\, G_F m_e E}{\Delta m^2}$$

Calculate $P(\nu_e \to \nu_\mu, t)$: same steps as vacuum

$$|\nu(t)\rangle = c_m e^{-iE_{1m}t} |\nu_{1m}\rangle + s_m e^{-iE_{2m}t} |\nu_{2m}\rangle$$

$$P(\nu_e \to \nu_\mu, t) = \left| \left(-s_m \langle \nu_{1m}| + c \langle \nu_{2m}| \right) |\nu(t)\rangle \right|^2$$

$$\vdots \quad (\text{steps})$$

$$\Rightarrow \boxed{P(\nu_e \to \nu_\mu, L) = \sin^2 2\vartheta_m \sin^2\left(\pi \frac{L}{\ell_m}\right)}$$

$$\boxed{\ell_m = \frac{2\pi}{E_{2m} - E_{1m}} = \frac{\ell_0}{\left[\sin^2 2\vartheta + (\cos 2\vartheta - x)^2\right]^{1/2}}} \qquad \left(\ell_0 = \frac{4\pi E}{\Delta m^2}\right)$$

$$H_{fl} = \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos 2\vartheta + x & \sin 2\vartheta \\ \sin 2\vartheta & \cos 2\vartheta - x \end{pmatrix}$$

$$x \equiv \frac{2\sqrt{2}\, G_F\, n_e\, E}{\Delta m^2}$$

$$\begin{cases} \sin 2\vartheta_m = \dfrac{\sin 2\vartheta}{\left[\sin^2 2\vartheta + (\cos 2\vartheta - x)^2\right]^{1/2}} \\[2em] \cos 2\vartheta_m = \dfrac{\cos 2\vartheta - x}{\left[\sin^2 2\vartheta + (\cos 2\vartheta - x)^2\right]^{1/2}} \end{cases}$$

$$E_{2m} - E_{1m} = \frac{\Delta m^2}{2E} \left[\sin^2 2\vartheta + (\cos 2\vartheta - x)^2\right]^{1/2}$$

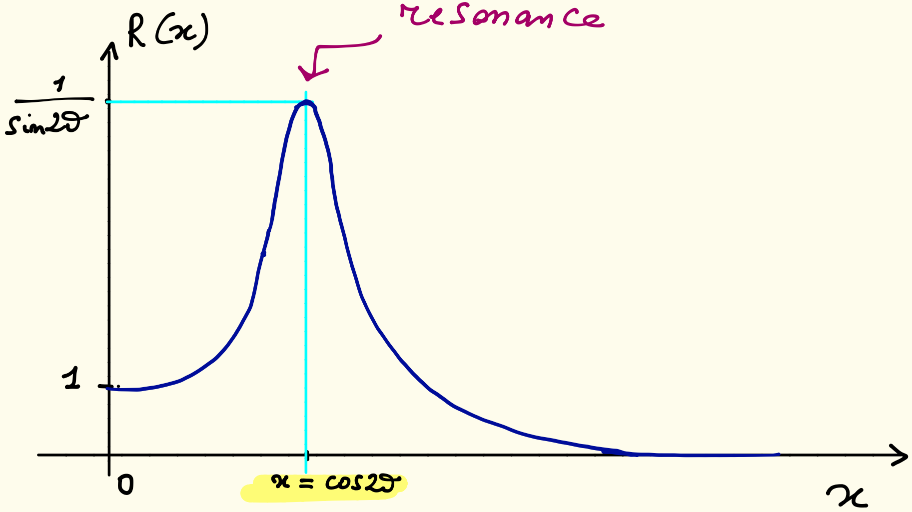
$$\ell_m = \frac{2\pi}{E_{2m} - E_{1m}} = \frac{\ell_0}{\left[\sin^2 2\vartheta + (\cos 2\vartheta - x)^2\right]^{1/2}}$$

$$P(\nu_e \to \nu_\mu, L) = \sin^2 2\vartheta_m \, \sin^2\left(\pi \frac{L}{\ell_m}\right)$$

Note :

- Vacuum solution recovered for $x=0$ ($n_e = 0$), as should be.

- <u>Resonant character</u> : factor $\dfrac{1}{\left[\sin^2 2\vartheta + (\cos 2\vartheta - x)^2\right]^{1/2}} \equiv R(x)$



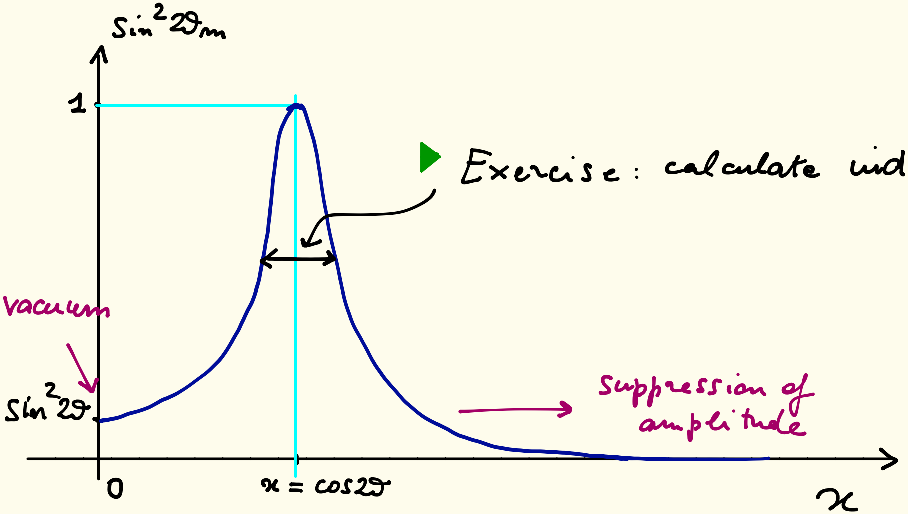
At resonance:

$$\boxed{\sqrt{2}\, G_F n_e = \frac{\Delta m^2}{2E} \cos 2\vartheta}$$ (MSW resonance condition)

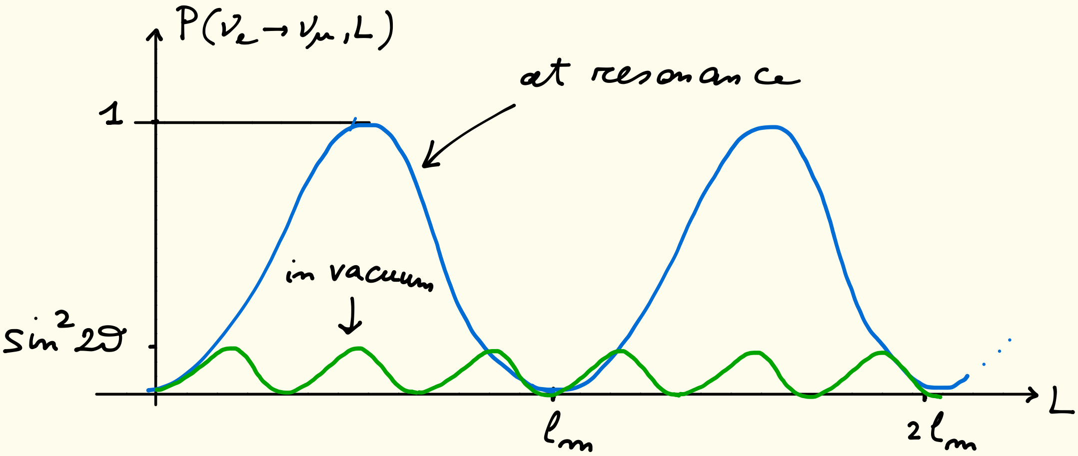
- $E_{1m} - E_{2m} \propto R(x)^{-1}$ has a minimum

- $\ell_m \propto R(x)$ has a maximum

- $\sin^2 2\vartheta_m \propto R(x)^2$ has a maximum $\longrightarrow$ <u>max amplitude of oscill.</u>



▶ Exercise: calculate width at ½ max.



- Useful : resonance energy, resonance density ($\rho = 2 m_N n_e$)

$$E_R = \frac{\Delta m^2 \cos 2\vartheta}{2\sqrt{2}\, G_F n_e} = 14\ \mathrm{GeV} \left(\frac{\Delta m^2}{10^{-3}\,\mathrm{eV}^2}\right)\left(\frac{1\ \mathrm{g\,cm^{-3}}}{\rho}\right) \cos 2\vartheta$$

$$n_{eR} = \frac{\Delta m^2 \cos 2\vartheta}{2\sqrt{2}\, G_F E}$$

$$\hookrightarrow \rho_R = 2 m_N n_{eR} = 1.4\ \mathrm{g\,cm^{-3}} \cos 2\vartheta \left(\frac{\Delta m^2}{10^{-3}\,\mathrm{eV}^2}\right)\left(\frac{10\ \mathrm{GeV}}{E}\right)$$

- Resonance exists only if $\Delta m^2 > 0$!

  For $\Delta m^2 < 0$ $\sin^2 2\theta_m \leq \sin^2 2\theta$ $\longrightarrow$ amplitude always suppressed.

- for antineutrinos: $V_\alpha \longrightarrow -V_\alpha < 0$

  Resonance for $\Delta m^2 < 0$, suppression for $\Delta m^2 > 0$.

  Resonance affects only ν <u>OR</u> $\bar{\nu}$, not both.

2.2) Varying density: $\rho = \rho(t)$, 2×2 case

$H = H(t) \longrightarrow$ need full Schroedinger equation.

Idea: use basis of instantaneous eigenstates:

$|\nu_{1m}(t)\rangle$, $|\nu_{2m}(t)\rangle$, $\vartheta_m = \vartheta_m(t)$

$$\begin{cases} |\nu_e\rangle = c_m(t)\,|\nu_{1m}(t)\rangle + s_m(t)\,|\nu_{2m}(t)\rangle \\ |\nu_\mu\rangle = -s_m(t)\,|\nu_{1m}(t)\rangle + c_m(t)\,|\nu_{2m}(t)\rangle \end{cases} \longrightarrow U_m = \begin{pmatrix} c_m & s_m \\ -s_m & c_m \end{pmatrix}$$

$c_m(t) \equiv \cos\vartheta_m(t)$, $s_m(t) = \sin\vartheta_m(t)$

$$\boxed{U(\vartheta_m(t)) = U_m(t)}$$

Schroedinger equation, flavor basis:

$$i \frac{d\nu_{fl}}{dt} = H_{fl}\, \nu_{fl} \qquad\qquad \nu_{fl} = U_m(t)\, \nu_m$$

Rewrite in instantaneous matter basis:

$$i \frac{d}{dt}\left(U_m(t)\, \nu_m(t) \right) = H_{fl} \left(U_m(t)\, \nu_m(t) \right)$$

$$i \left(\frac{d}{dt} U_m \right) \nu_m + i\, U_m \frac{d}{dt} \nu_m = H_{fl}\, U_m\, \nu_m$$

$$\downarrow \times U_m^{+} :$$

$$i \frac{d}{dt} \nu_m = \left[U_m^{+} H_{fl}\, U_m - \underbrace{i\, U_m^{+} \left(\frac{d}{dt} U_m \right)}_{\text{depends on } \dot{\vartheta}_m = \frac{d}{dt} \vartheta_m} \right] \nu_m$$

Result, explicit form:

$$i\frac{d}{dt}\begin{pmatrix} \nu_{1m} \\ \nu_{2m} \end{pmatrix} = \begin{pmatrix} E_{1m}(t) & -i\dot{\vartheta}_m(t) \\ i\dot{\vartheta}_m(t) & E_{2m}(t) \end{pmatrix}\begin{pmatrix} \nu_{1m} \\ \nu_{2m} \end{pmatrix}$$

Note: off-diagonal terms $\propto \dot{\vartheta}_m \longrightarrow$ cause $\nu_{1m} \longleftrightarrow \nu_{2m}$ transitions

Adiabatic approximation: neglect $\dot{\vartheta}_m$

Adiabaticity condition:

$$|\dot{\vartheta}_m| \ll |E_{1m} - E_{2m}|$$

$$\gamma^{-1} \equiv \frac{2|\dot{\vartheta}_m|}{|E_{1m}-E_{2m}|} = \frac{\sin 2\vartheta \; \Delta m^2/2E}{|E_{1m}-E_{2m}|^3} |\dot{V}_{cc}| \ll 1$$

Meaning: potential varies slowly

$\longrightarrow$ $\nu_{1m} \leftrightarrow \nu_{2m}$ transitions suppressed

$\longrightarrow$ $\langle \nu_{1m} | \nu \rangle$, $\langle \nu_{2m} | \nu \rangle$ vary only by a phase

Apply adiabatic approximation:

t_i : initial time

t_f : final time

$$|\nu(t_i)\rangle = |\nu_e\rangle = \cos\theta_{mi}\,|\nu_{1m}\rangle + \sin\theta_{mi}\,|\nu_{2m}\rangle$$

$$|\nu(t_f)\rangle = \cos\theta_{mi}\, e^{-i\int_{t_i}^{t_f} E_{1m}(t')dt'}\,|\nu_{1m}\rangle + \sin\theta_{mi}\, e^{-i\int_{t_i}^{t_f} E_{2m}(t')dt'}\,|\nu_{2m}\rangle$$

at $t=t_f$: $|\nu_\mu\rangle = -\sin\theta_{mf}\,|\nu_{1m}\rangle + \cos\theta_{mf}\,|\nu_{2m}\rangle$

$$P(\nu_e\to\nu_\mu, t) = \left|\langle\nu_\mu|\nu(t_f)\rangle\right|^2$$

Full result:

$$P(\nu_e \to \nu_\mu, t) = \frac{1}{2} - \frac{1}{2}\cos 2\theta_{mi} \cos 2\theta_{mf} - \frac{1}{2}\sin 2\theta_{mi} \sin 2\theta_{mf} \cos\varphi$$

$$\varphi = \int_{t_i}^{t_f} \Big(E_{1m}(t') - E_{2m}(t') \Big) dt'$$

If $\sin 2\theta_{mi} \simeq 0$ OR $\sin 2\theta_{mf} \simeq 0$ or $\langle \cos\varphi \rangle \simeq 0$
(averaged oscill.):

$$\boxed{P(\nu_e \to \nu_\mu, t) = \frac{1}{2} - \frac{1}{2}\cos 2\theta_{mi} \cos 2\theta_{f}}$$

Common situation : $m_e(t_i) \gg m_{eR} \longrightarrow \cos 2\vartheta_{m_i} \simeq -1$

$m_e(t_f) \ll m_{eR} \longrightarrow \cos 2\vartheta_{m_f} \simeq \cos 2\vartheta$

$\hookrightarrow P(\nu_e \to \nu_\mu, t) \simeq \frac{1}{2}(1+\cos 2\vartheta)$

$= \cos^2\vartheta \geq \frac{1}{2}$

strong flavor conversion!

$(\vartheta < \pi/4)$

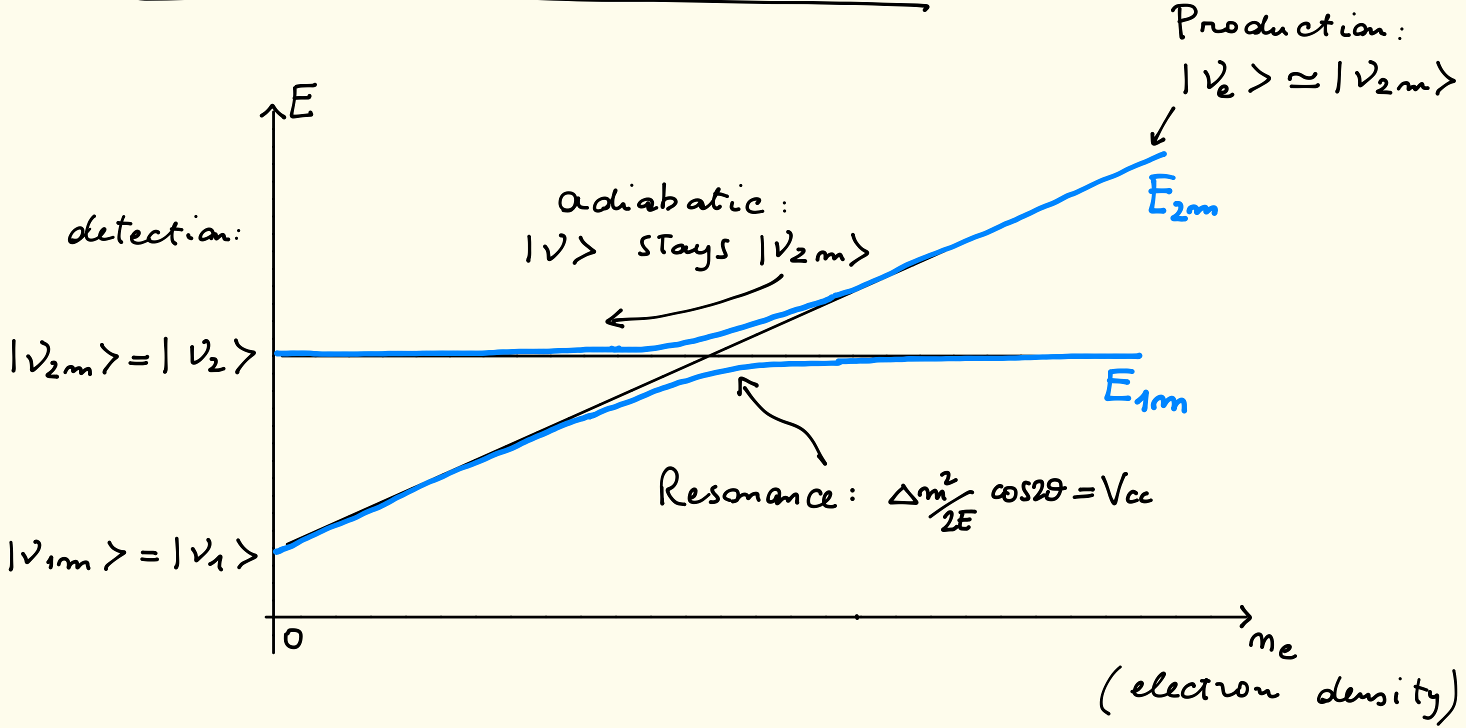
Qualitative picture:

at $t = t_i$, $|\nu_e\rangle \sim |\nu_{2m}\rangle$

adiabatic evolution: $|\nu_{2m}\rangle \longrightarrow |\nu_2\rangle$ (up to a phase)

at t_f : $P(\nu_e \to \nu_\mu) = |\langle \nu_\mu | \nu_2 \rangle|^2 = \cos^2\vartheta$

Useful: level-crossing diagram



A paradox ?

$$P(\nu_e \to \nu_\mu, t) \simeq \tfrac{1}{2}(1+\cos 2\vartheta) \xrightarrow{\vartheta \to 0} 1$$

Mixing-less conversion ???

No, for $\vartheta \to 0$ adiabatic condition is violated!

Generalization to non-adiabatic propagation (for $\vartheta \ll \pi/4$)

$$\gamma^{-1} \equiv \frac{2|\dot{\vartheta}_m|}{|E_{1m}-E_{2m}|} = \frac{\sin 2\vartheta \; \Delta m^2/2E}{|E_{1m}-E_{2m}|^3} \, |\dot{V}_{cc}|$$

<u>At resonance</u>: $|E_{1m}-E_{2m}|$ is min. $\longrightarrow$ γ^{-1} is max.

$\longrightarrow$ max probability of $\nu_{1m} \longleftrightarrow \nu_{2m}$

γ at resonance:

$$\gamma_r = \frac{\sin^2 2\vartheta}{\cos 2\vartheta} \, \frac{\Delta m^2}{2E} \left| \frac{\dot{n}_e}{n_e} \right|_{res}^{-1} \xrightarrow{\vartheta \to 0} 0$$

max. violation of adiabaticity

Generalized probability (oscill. averaged):

$$\boxed{P(\nu_e \to \nu_\mu, t) = 1/2 - \frac{1}{2} \cos 2\vartheta_{mi} \cos 2\vartheta_{mf} \left(1 - 2 P_{LZ} \right)}$$

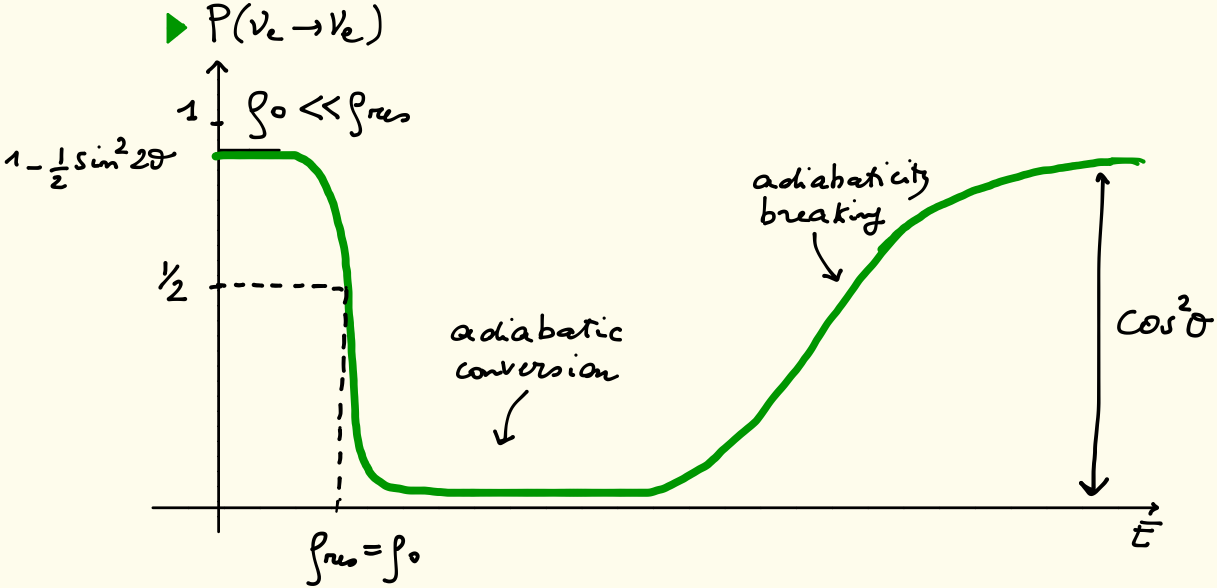
$$\boxed{P_{LZ} = e^{-\pi/2\, \gamma_r}}$$

Landau-Zener hopping probability ($\vartheta \ll \pi/4$)

Useful exercise: $P(\nu_e \to \nu_e)$ as function of E (oscill. averaged)

Example: Sun: $\rho(r) \simeq \rho_0 \, e^{-r/R_\odot}$ $\qquad \rho_0 \sim 10^2 \, g\,cm^{-3}$

Take $\theta \ll \pi/4$ (small mixing)



Beyond basics: 3 ν in vacuum

$\{\nu_1, \nu_2, \nu_3\}$, $\{m_1, m_2, m_3\}$ $\{\nu_e, \nu_\mu, \nu_\tau\}$

Mixing matrix:

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

$\theta_{12}, \theta_{23}, \theta_{13}$ $c_{ij} = \cos\theta_{ij}$ $s_{ij} = \sin\theta_{ij}$

[Note: Majorana phases do not affect oscillations]

A note on notation

$\nu_\alpha \equiv \langle \nu_\alpha | \nu \rangle$ component of flavor $\alpha = e, \mu, \tau$

$\nu_\alpha = U_{\alpha j}\, \nu_j$ (summed over $j = 1, 2, 3$) :

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

Instead :

$$| \nu_\alpha \rangle = U^*_{\alpha j} | \nu_j \rangle$$

Calculate probabilities : Vacuum case

$$P(\nu_\alpha \to \nu_\beta, t) = |A(\nu_\alpha \to \nu_\beta)|^2$$

$$A(\nu_\alpha \to \nu_\beta) = \left(\sum_{i,j} \langle \nu_i | U_{\beta i}\right)\left(\sum_i e^{-iE_j t} U^*_{\alpha j} |\nu_j\rangle\right)$$

⋮ (steps)

$$P(\nu_\alpha \to \nu_\beta, t) = \sum_{i,j} U^*_{\alpha j} U_{\beta j} U^*_{\beta i} U_{\alpha i} e^{-i(E_j - E_i)t}$$

Effective 2ν descriptions : use $\Delta m^2_{21} \ll |\Delta m^2_{31}| \simeq |\Delta m^2_{32}|$

1) $\frac{\Delta m^2_{21}}{2E} L \ll 1$ $\longrightarrow$ take $\Delta m^2_{21} = 0$

$$\boxed{P(\nu_e \to \nu_e, L) \simeq 1 - \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2_{31}}{4E} L\right)}$$

(ν beams)

2) $\left|\frac{\Delta m^2_{31}}{2E} L\right| \sim \left|\frac{\Delta m^2_{32}}{2E} L\right| \gg 1 \rightarrow 3\to1,\ 3\to2$ oscill. average out

$$\boxed{P(\nu_e \to \nu_e, L) \simeq S^2_{13} + C^2_{13}\, P_{2\nu}(\Delta m^2_{21}, \theta_{12})}$$

(reactor ν)

Exact solution of 3×3 case :

$$P(\nu_\alpha \to \nu_\beta, t) = \sum_{i,j} U^*_{\alpha j} U_{\beta j} U^*_{\beta i} U_{\alpha i} \, e^{-i(E_j - E_i)t}$$

⋮ (steps, use unitarity)

$$\boxed{\begin{aligned} P(\nu_\alpha \to \nu_\beta, L) = \delta_{\alpha\beta} &- 4 \sum_{j>i} \mathrm{Re}\left[U^*_{\alpha j} U_{\beta j} U^*_{\beta i} U_{\alpha i} \right] \sin^2\left(\frac{\Delta m^2_{ij}}{4E} L \right) \\ &+ 2 \sum_{j>i} \mathrm{Im}\left[U^*_{\alpha j} U_{\beta j} U^*_{\beta i} U_{\alpha i} \right] \sin\left(\frac{\Delta m^2_{ij}}{2E} L \right) \end{aligned}}$$

CP symmetry on oscillation probabilities

$\nu_{\alpha L} = U_{\alpha j}\, \nu_{jL}$: $P(\nu_{\alpha L} \longrightarrow \nu_{\beta L}, t)$ ✓

$\big\downarrow C$

C : $\bar{\nu}_{\alpha L} = U^*_{\alpha j}\, \bar{\nu}_{jL}$ $P(\bar{\nu}_{\alpha L} \longrightarrow \bar{\nu}_{\beta L}, t)$ ✗ not observed!

(charge conjugation)

$\big\downarrow P$

CP : $\bar{\nu}_{\alpha R} = U^*_{\alpha j}\, \bar{\nu}_{jR}$ $P(\bar{\nu}_{\alpha R} \longrightarrow \bar{\nu}_{\beta R}, t)$ ✓

(charge and parity)

$\Leftrightarrow$ CP symmetry : $\boxed{P(\nu_\alpha \rightarrow \nu_\beta) = P(\bar{\nu}_\alpha \longrightarrow \bar{\nu}_\beta)}$

Look for CP violation :

calculate $P(\nu_\alpha \to \nu_\beta) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) \equiv \Delta P_{\alpha\beta}$

$$P(\nu_\alpha \to \nu_\beta, L) = \delta_{\alpha\beta} - 4 \sum_{j>i} \mathrm{Re}\left[U_{\alpha j}^* U_{\beta j} U_{\beta i}^* U_{\alpha i} \right] \sin^2\left(\frac{\Delta m_{ij}^2}{4E} L \right)$$

$$+ 2 \sum_{j>i} \mathrm{Im}\left[U_{\alpha j}^* U_{\beta j} U_{\beta i}^* U_{\alpha i} \right] \sin\left(\frac{\Delta m_{ij}^2}{2E} L \right)$$

$$P(\bar{\nu}_\alpha \to \bar{\nu}_\beta, L) = \delta_{\alpha\beta} - 4 \sum_{j>i} \mathrm{Re}\left[U_{\alpha j}^* U_{\beta j} U_{\beta i}^* U_{\alpha i} \right] \sin^2\left(\frac{\Delta m_{ij}^2}{4E} L \right)$$

$$- 2 \sum_{j>i} \mathrm{Im}\left[U_{\alpha j}^* U_{\beta j} U_{\beta i}^* U_{\alpha i} \right] \sin\left(\frac{\Delta m_{ij}^2}{2E} L \right)$$

$$\boxed{\Delta P_{\alpha\beta} = 4 \sum_{j>i} \mathrm{Im}\left[U_{\alpha j}^* U_{\beta j} U_{\beta i}^* U_{\alpha i} \right] \sin\left(\frac{\Delta m_{ij}^2}{2E} L \right)}$$

CP violation in the lepton sector ?

$$\boxed{\Delta P_{e\mu} = \Delta P_{\mu\tau} = \Delta P_{e\tau} = 4\, s_{12} c_{12} s_{13} c_{13}^2 s_{23} c_{23} \sin\delta \sum_{j>i} \sin\left(\frac{\Delta m_{ij}^2}{2E} L\right)}$$

CP is violated if

$$\Delta P_{\alpha\beta} \neq 0 \longrightarrow \boxed{\sin\delta \neq 0}$$

Note :

$$\Delta P_{\alpha\beta} = 0 \quad \text{if any } \vartheta_{ij} = 0 \longrightarrow \cancel{CP} \text{ is a } 3\nu \text{ effect!}$$

3ν propagation in matter: (outline only)

- $V_{\bar{\alpha}} = -V_\alpha \longrightarrow P(\nu_\alpha \to \nu_\beta) \neq P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) \longrightarrow$ mimics effect of δ

  consequence of CP-asymmetric background $(n_e \neq n_{\bar{e}})$

- generalized MSW: factorization in multiple 2ν problems

For further study: ν in ν background

- $\nu - \nu$ scattering : non-linear effects
  relevant for supernovae, cosmology

  [e.g. Duan & Kneller, arxiv: 0904.0974]