2014 National Nuclear Physics Summer School

WILLIAM & MARY

Lectures on Effective Field Theory

I. Removing heavy particles
II. Removing large scales
III. Describing Goldstone bosons
IV. Interacting with Goldstone bosons

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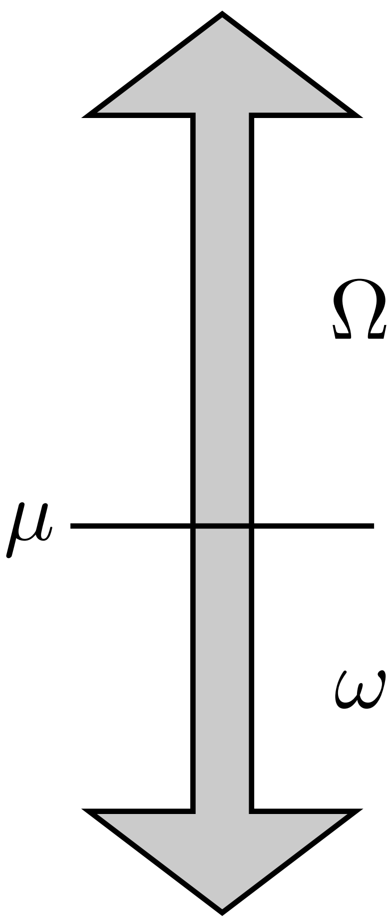
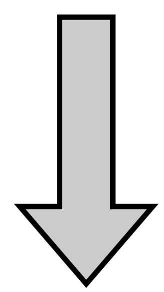
Effective Recap

- **“Top Down” EFTs**: systematically integrate out heavy particles, large energy scales



- Starting point: perturbative QFT (or EFT)



Power corrections $(\omega/\Omega)^n$

Perturbative corrections $\alpha_s(\Omega)$

... large logs

- End point: lower-scale EFT

Captures non-analyticities of effective d.o.f.

Designed to reproduce S-matrix elements, “matching”

- Greatly facilitates computations in energy regimes for which the full theory is cumbersome and unnecessary

Effective Field Theory

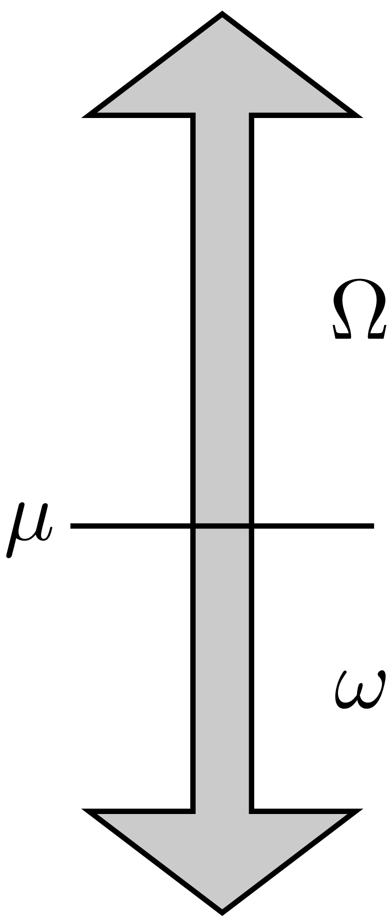
III. Describing Goldstone bosons

“Bottom Up” EFTs

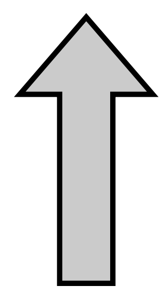
- Effective d.o.f. arise non-perturbatively

CMT: quasi-particles

QCD: low-lying hadrons



- End point: breakdown of EFT at higher-scale



Power corrections $(\omega/\Omega)^n$

Non-analytic corrections $\log \frac{\omega}{\Omega}$

- Starting point: low-scale EFT

Captures non-analyticities of IR d.o.f.

Matching with experiment / non-perturbative calculation

- Focus: spontaneous symmetry breaking for which EFT is constructed to account for Goldstone modes. **QCD** *chiral symmetry & pions*

Massless QCD

$$\mathcal{L}_{\rm QCD} = \mathcal{L}_\psi + \mathcal{L}_{YM}$$

$$\mathcal{L}_\psi = \sum_{i=1}^{N_f} \overline{\psi}_i i \not{D}\, \psi_i$$

- Take two flavors. These will correspond to up and down quarks.
- Massless QCD Lagrange density obviously has global U(2) flavor symmetry but...

Chiral symmetry

$$\mathcal{P}_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$$ projectors

$$\psi_{L,R} = \mathcal{P}_{L,R}\,\psi$$

$$\overline{\psi} i \not{D}\, \psi = \overline{\psi}_L i \not{D}\, \psi_L + \overline{\psi}_R i \not{D}\, \psi_R$$

- Left- and right-handed fields do not mix: no chirality changing interaction

$$U(2)_L \otimes U(2)_R$$

$L \qquad R$

$$\psi_L \to L\psi_L$$
$$\psi_R \to R\psi_R$$

$$\psi \to (L\mathcal{P}_L + R\,\mathcal{P}_R)\,\psi$$

Massless QCD

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Vector subgroup

~~$U(2)_L \otimes U(2)_R$~~ $\quad U(2)_V$

$L = R = V$

$$\psi_L \to V\psi_L$$
$$\psi_R \to V\psi_R$$

$$\psi \to V(\mathcal{P}_L + \mathcal{P}_R)\psi = V\psi$$

Chiral Symmetry of Massless QCD

Action invariant under $U(2)_L \otimes U(2)_R = U(1)_L \otimes U(1)_R \otimes SU(2)_L \otimes SU(2)_R$

$$U(1)_L : \psi_L \to e^{i\theta_L}\psi_L$$
$$U(1)_R : \psi_R \to e^{i\theta_R}\psi_R$$

$$\psi \to \left[\frac{1}{2}(e^{i\theta_R} + e^{i\theta_L}) + \frac{1}{2}(e^{i\theta_R} - e^{i\theta_L})\gamma_5\right]\psi$$

Vector subgroup $\theta_L = \theta_R = \theta$ $\quad \psi \to e^{i\theta}\psi$ $\quad U(1)_V$

Axial transformation $-\theta_L = \theta_R = \theta_5$ $\quad \psi \to [\cos\theta_5 + i\gamma_5 \sin\theta_5]\psi = e^{i\theta_5\gamma_5}\psi$

$U(1)_A$

Exercise:

Consider a non-singlet axial transformation $\psi_i \to [\exp(i\vec{\phi}\cdot\vec{\tau}\gamma_5)]_{ij}\psi_j$

Is there a corresponding symmetry group of the massless QCD action?

Chiral Symmetry of Massless QCD

Action invariant under $U(2)_L \otimes U(2)_R = U(1)_A \otimes U(1)_V \otimes SU(2)_L \otimes SU(2)_R$

- Global symmetries lead to *classically* conserved currents

$$J_{5\mu} = \overline{\psi}\gamma_\mu\gamma_5\psi \qquad J_\mu = \overline{\psi}\gamma_\mu\psi \qquad J^a_{L\mu} = \overline{\psi}_L\gamma_\mu\tau^a\psi_L \qquad J^a_{R\mu} = \overline{\psi}_R\gamma_\mu\tau^a\psi_R$$

(Regulated) Theory not invariant under flavor-singlet axial transformation

$$\partial^\mu J_{5\mu}(x) = \partial^\mu J_{R\mu}(x) - \partial^\mu J_{L\mu}(x) = \begin{cases} \frac{e}{2\pi} N_f\, \epsilon^{\mu\nu} F_{\mu\nu}(x) & d=2 \\ \frac{\alpha_s}{4\pi} N_f\, \epsilon^{\mu\nu\rho\sigma} G^A_{\mu\nu}(x) G^A_{\rho\sigma}(x) & d=4 \end{cases}$$

Chiral Anomaly

$U(1)_V \otimes SU(2)_L \otimes SU(2)_R$

The chiral anomaly obstructs chirally invariant lattice regularization of fermions (**see Lattice QCD lectures**)

Fate of Symmetries in Low-Energy QCD

$$U(1)_V \otimes SU(2)_L \otimes SU(2)_R$$

- Chiral pairing preferred by vacuum (non-perturbative ground state)

Chiral condensate $\langle\overline{\psi}\psi\rangle = \langle\overline{\psi}_R\psi_L\rangle + \langle\overline{\psi}_L\psi_R\rangle \neq 0$

- Massless quarks *can* change their chirality by scattering off vacuum condensate

$$U(1)_V \otimes SU(2)_L \otimes SU(2)_R \longrightarrow U(1)_V \otimes SU(2)_V$$

- Spontaneously broken symmetries lead to massless bosonic excitations

Nambu-Goldstone Mechanism

Broken generators in coset $SU(2)_L \otimes SU(2)_R / SU(2)_V$

Number of massless particles?

Chiral Condensate

$$\langle \overline{\psi}_{iR}\psi_{jL}\rangle = -\lambda\, \delta_{ji}$$

- Choice for vacuum orientation $\lambda \in \mathbb{R}$ from Vafa-Witten (P)
- After a chiral transformation $\langle \overline{\psi}_{iR}\psi_{jL}\rangle \to L_{jj'}R^{\dagger}_{i'i}\langle \overline{\psi}_{i'R}\psi_{j'L}\rangle = -\lambda(LR^{\dagger})_{ji}$

$$SU(2)_L \otimes SU(2)_R \longrightarrow SU(2)_V$$

- Describe Goldstone fluctuations of vacuum state with fields

$$\delta_{ji} \to \Sigma_{ji}(x) = \delta_{ji} + \ldots \qquad \Sigma \in SU(2)_L \otimes SU(2)_R / SU(2)_V$$

$$[L(x)R^{\dagger}(x)]_{ji} = [e^{i\vec{\theta}_L(x)\cdot\vec{\tau}}e^{-i\vec{\theta}_R(x)\cdot\vec{\tau}}]_{ji} \qquad \vec{\theta}_L = -\vec{\theta}_R$$

$$\Sigma = e^{2i\phi/f} = 1 + \frac{2i\phi}{f} + \ldots$$

Transformation properties

$$\Sigma \to L\Sigma R^{\dagger} \qquad \Sigma \to V\Sigma V^{\dagger} \qquad \phi \to V\phi V^{\dagger}$$

Exercise:

Determine the discrete symmetry properties of the Goldstone modes from the coset's transformation.

Dynamics of Goldstone Bosons: Chiral Lagrangian

$$\Sigma = e^{2i\phi/f} = 1 + \frac{2i\phi}{f} + \ldots \qquad \mathrm{Tr}\,\phi = 0 \qquad \phi^\dagger = \phi$$

The Pions $\phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 & \pi^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 \end{pmatrix}$

- Build chirally invariant theory of coset field

$$\Sigma \to L\Sigma R^\dagger \qquad \Sigma^\dagger \to R\Sigma^\dagger L \qquad \Sigma^\dagger\Sigma = 1$$

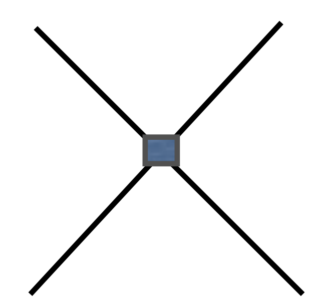
$$\mathcal{L} = \frac{f^2}{8}\mathrm{Tr}\left(\partial^\mu\Sigma\partial_\mu\Sigma^\dagger\right)$$

- Expand about v.e.v. to uncover Gaussian fluctuations

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}\,\left(\partial^\mu\phi\,\partial_\mu\phi\right) + \mathcal{O}(1/f^2) = \frac{1}{2}\partial^\mu\pi^0\partial_\mu\pi^0 + \partial^\mu\pi^-\partial_\mu\pi^+ + \mathcal{O}(1/f^2)$$

3 massless modes

- Non-linear theory: interactions between multiple pions at “higher orders”



Can treat systematically...

Including Quark Masses

- We began with massless QCD. Quarks have mass, Higgs makes two very light
- Chiral symmetry of action is only approximate: explicit symmetry breaking

$$\Delta\mathcal{L}_\psi = -m_q \sum_i \overline{\psi}_i \psi_i = -m_q \sum_i \left(\overline{\psi}_{iR}\psi_{iL} + \overline{\psi}_{iL}\psi_{iR} \right)$$

$SU(2)_L \otimes SU(2)_R \longrightarrow SU(2)_V$ $\qquad$ $m_q/\Lambda_{\rm QCD} \ll 1$

- Need to map $\Delta\mathcal{L}_\psi$ onto ChPT operators breaking symmetry in same way

$$\Delta\mathcal{L}_{\rm eff} = m_q \lambda \, {\rm Tr}\left(\Sigma + \Sigma^\dagger\right)$$

Comments: not chirally invariant $\qquad$ $\Sigma \rightarrow L\Sigma R^\dagger$, $\Sigma^\dagger \rightarrow R\Sigma^\dagger L^\dagger$

new dimensionful parameter $\qquad$ λ

included only linear quark mass term $\qquad$ m_q^2

Perturbing about chiral limit

Chiral Lagrangian

$$\mathcal{L}_\chi = \frac{f^2}{8}\mathrm{Tr}\left(\partial^\mu\Sigma\partial_\mu\Sigma^\dagger\right) + m_q\lambda\,\mathrm{Tr}\left(\Sigma+\Sigma^\dagger\right)$$

- Expand up to quadratic order $\mathcal{L}_\chi = 4m_q\lambda + \frac{1}{2}\mathrm{Tr}\left(\partial^\mu\phi\,\partial_\mu\phi\right) - \frac{8m_q\lambda}{f^2}\frac{1}{2}\mathrm{Tr}\left(\phi\phi\right)$

  Pion mass $m_\pi^2 = 8m_q\lambda/f^2$

- Vacuum energy must be due to chiral condensate (ingredient in our construction)

  QCD degrees of freedom

$$Z_{\mathrm{QCD}}[m_q,\ldots] = \int\mathcal{D}\cdots e^{i\int_x(\cdots - m_q\overline{\psi}\psi)}$$

$$i\frac{\partial\log Z_{\mathrm{QCD}}}{\partial m_q} = \langle\overline{\psi}\psi\rangle$$

Low-energy degrees of freedom

Effective field theory

$$Z_{\chi\mathrm{PT}}[m_q,\ldots] = \int\mathcal{D}\Sigma\, e^{i\int_x\mathcal{L}_\chi(\Sigma;m_q)}$$

$$Z_{\mathrm{QCD}}[m_q,\ldots] \equiv Z_{\chi\mathrm{PT}}[m_q,\ldots]$$

Matching

$$\langle\overline{\psi}\psi\rangle = i\frac{\partial\log Z_{\chi\mathrm{PT}}}{\partial m_q} = -\lambda\langle\mathrm{Tr}\left(\Sigma+\Sigma^\dagger\right)\rangle = -2N_f\lambda$$

From before:

$$\langle\overline{\psi}_{iR}\psi_{jL}\rangle = -\lambda\,\delta_{ji}$$

$$\lambda = \lambda$$

Chiral Lagrangian

$$\mathcal{L}_\chi = \frac{f^2}{8}\mathrm{Tr}\left(\partial^\mu \Sigma \partial_\mu \Sigma^\dagger\right) + m_q \lambda\, \mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right)$$

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Pion mass $m_\pi^2 = 8m_q\lambda/f^2$ $\qquad f^2 m_\pi^2 = 2m_q|\langle\overline{\psi}\psi\rangle|$ (Gell-Mann Oakes Renner)

- Vacuum energy must be due to chiral condensate (ingredient in our construction)

QCD degrees of freedom

$$Z_{\mathrm{QCD}}[m_q,\ldots] = \int \mathcal{D}\cdots e^{i\int_x(\cdots - m_q\overline{\psi}\psi)}$$

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Low-energy degrees of freedom

Effective field theory

$$Z_{\chi\mathrm{PT}}[m_q,\ldots] = \int \mathcal{D}\Sigma\, e^{i\int_x \mathcal{L}_\chi(\Sigma;\,m_q)}$$

$$Z_{\mathrm{QCD}}[m_q,\ldots] \equiv Z_{\chi\mathrm{PT}}[m_q,\ldots]$$

Matching

$$\langle\overline{\psi}\psi\rangle = i\frac{\partial \log Z_{\chi\mathrm{PT}}}{\partial m_q} = -\lambda\langle\mathrm{Tr}\left(\Sigma+\Sigma^\dagger\right)\rangle = -2N_f\lambda$$

From before:

$$\langle\overline{\psi}_{iR}\psi_{jL}\rangle = -\lambda\,\delta_{ji}$$

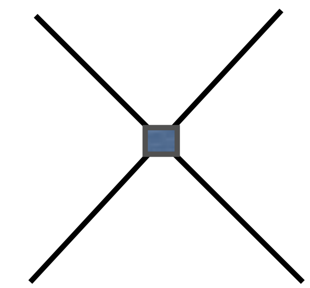
$$\lambda = \lambda$$

ChPT

$$\mathcal{L}_\chi = \frac{f^2}{8}\mathrm{Tr}\left(\partial^\mu\Sigma\partial_\mu\Sigma^\dagger\right) + m_q\lambda\,\mathrm{Tr}\left(\Sigma+\Sigma^\dagger\right)$$

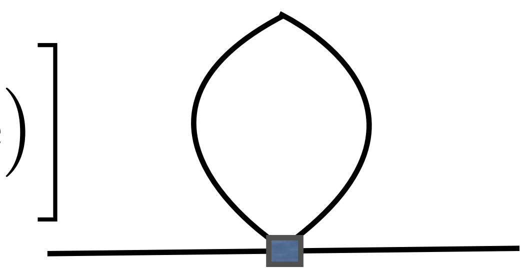
- Quadratic fluctuations are the approximate Goldstone bosons of SChSB
- Quartic terms describe interactions

$$\sim \frac{m_q\lambda}{f^4}\phi^4 \quad \sim \frac{1}{f^2}(\phi\partial_\mu\phi)^2$$



- Higher-order interactions renormalize lower-order terms

$$\Delta m_\pi^2 \sim \frac{m_q\lambda}{f^4}\int_k \frac{i}{k^2-m_\pi^2} \sim \frac{m_q\lambda}{f^2}\left[\Lambda^2 + \frac{m_q\lambda}{f^2}\left(\log\Lambda^2 + \mathrm{finite}\right)\right]$$



power-law divergence

Absorb in renormalized mass,
or just use dimensional regularization

logarithmic divergence

Renormalization requires new
operator in chiral Lagrangian

- ChPT is non-renormalizable (needing infinite local terms to renormalize)

  1). low-energy theory, so who cares?

  2). must be able to order terms in terms of relevance "power counting"

Power Counting

$$\mathcal{L}_\chi = \frac{f^2}{8}\mathrm{Tr}\left(\partial^\mu\Sigma\partial_\mu\Sigma^\dagger\right) + m_q\lambda\,\mathrm{Tr}\left(\Sigma+\Sigma^\dagger\right)$$

- Leading-order Lagrangian in expansion in derivatives and quark mass

$\mathcal{O}(p^2)$ $\partial_\mu \sim p$ $m_q \sim p^2$ Low-energy dynamics of pions

Propagator $\frac{i}{k^2 - m_\pi^2} \sim p^{-2}$ Vertices $\partial^\mu\partial_\mu,\, m_q \sim p^2$ Loop integral $\int_k \sim p^4$

General Feynman diagram: L loops, I internal lines, V vertices $\sim p^{4L-2I+2V}$

Euler formula $L = I - V + 1$ $\sim p^{2L+2}$

- Loop expansion: one loop graphs require only $\mathcal{O}(p^4)$ counterterms

Two loop graphs?

$\mathcal{O}(p^4)$ Chiral Lagrangian

- Construct chirally invariant terms out of coset $\Sigma \to L \Sigma R^\dagger$
  $\Sigma^\dagger \to R \Sigma^\dagger L^\dagger$

E.g. $\mathcal{O}(p^2)$ Lagrangian $\partial^\mu \Sigma \, \partial_\mu \Sigma^\dagger$

- Construct terms that break chiral symmetry in the same way as mass term

Simplification: add external scalar field to QCD action $\Delta\mathcal{L} = -\overline{\psi}_L s \, \psi_R - \overline{\psi}_R s^\dagger \psi_L$

Make the scalar transform to preserve chiral symmetry $s \to L \, s \, R^\dagger$

Giving the scalar a v.e.v. breaks chiral symmetry just as a mass $s = m_q + \cdots$

E.g. $\mathcal{O}(p^2)$ Lagrangian $\Sigma s^\dagger + s \Sigma^\dagger$

$\mathcal{O}(p^4)$ Chiral Lagrangian

Also impose Euclidean invariance, C, P, T

$$\Sigma \to L\Sigma R^\dagger \qquad s \to L\, s\, R^\dagger$$
$$\Sigma^\dagger \to R\Sigma^\dagger L^\dagger \qquad s^\dagger \to R s^\dagger L^\dagger$$

E.g. $[\mathrm{Tr}\left(\Sigma s^\dagger - s\Sigma^\dagger\right)]^2 \to m_q^2 [\mathrm{Tr}\left(\Sigma - \Sigma^\dagger\right)]^2 = 0$

Easy to generate terms. Care needed to find *minimal* set.

$$\mathcal{L}_4 = \mathbb{L}_1 [\mathrm{Tr}\left(\partial^\mu \Sigma \partial_\mu \Sigma^\dagger\right)]^2 + \mathbb{L}_2 \mathrm{Tr}\left(\partial^\mu \Sigma \partial^\nu \Sigma^\dagger\right) \mathrm{Tr}\left(\partial_\mu \Sigma \partial_\nu \Sigma^\dagger\right)$$
$$+ \mathbb{L}_3 \frac{m_q \lambda}{f^2} \mathrm{Tr}\left(\partial^\mu \Sigma \partial_\mu \Sigma^\dagger\right) \mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right) + \mathbb{L}_4 \frac{(m_q\lambda)^2}{f^4} [\mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right)]^2$$

$\{\mathbb{L}_j\}$ low-energy constants = Gasser-Leutwyler coefficients, dimensionless

N.B. these are not Gasser-Leutwyler**'s** coefficients

Complete set of counterterms needed to renormalize one-loop ChPT

Additional terms necessary when coupling external fields...

Exercise:

Determine the effects of strong isospin breaking $m_u \neq m_d$ on the chiral Lagrangian. At what order does the pion isospin multiplet split?

Simplest one-loop computation: Chiral Condensate

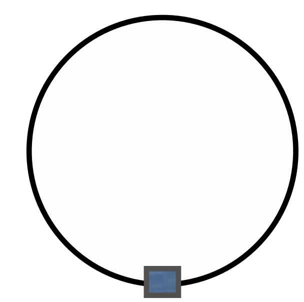
$$\langle\bar{\psi}\psi\rangle = i\frac{\partial \log Z_{\chi\mathrm{PT}}}{\partial m_q} = -\lambda\langle\mathrm{Tr}\,(\Sigma+\Sigma^\dagger)\rangle = -2N_f\lambda$$

"Tree-Level"

$$\Sigma+\Sigma^\dagger = 2 - \frac{4}{f^2}\phi^2 + \cdots$$

One Loop

$$\Delta\langle\bar{\psi}\psi\rangle = +\frac{4\lambda}{f^2}\times 3\,G_\pi(0) = \frac{12\lambda}{f^2}\int_k \frac{i}{k^2-m_\pi^2}$$



Dimensionally regulated integral $-\frac{m_\pi^2}{(4\pi)^2}\left(\frac{1}{\epsilon}-\gamma_E+\log 4\pi+\log\frac{\mu^2}{m_\pi^2}+1\right)$

$$-\mathbb{L}_3\frac{\lambda}{f^2}\mathrm{Tr}\,(\partial^\mu\Sigma\partial_\mu\Sigma^\dagger)\,\mathrm{Tr}\,(\Sigma+\Sigma^\dagger) - 2\mathbb{L}_4\frac{m_q\lambda^2}{f^4}[\mathrm{Tr}\,(\Sigma+\Sigma^\dagger)]^2$$

$$= -32\,\mathbb{L}_4\frac{m_q\lambda^2}{f^4} = -4\lambda\frac{m_\pi^2}{f^2}\mathbb{L}_4$$

$\mathcal{O}(p^4)$

"Tree-Level"

Final result: Chiral Logarithm

$$\langle\bar{\psi}\psi\rangle = -4\lambda\left[1+\frac{3\,m_\pi^2}{(4\pi f)^2}\left(\log\frac{\mu^2}{m_\pi^2}+1\right)+\frac{m_\pi^2}{f^2}\mathbb{L}_4(\mu)\right]$$

$$\mu^2\frac{d}{d\mu^2}\mathbb{L}_4 = -\frac{3}{16\pi^2}$$

Two-Flavor ChPT

$$\mathcal{L}_2 = \frac{f^2}{8}\left[\mathrm{Tr}\left(\partial^\mu \Sigma \partial_\mu \Sigma^\dagger\right) + m_q \lambda\, \mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right)\right]$$

- Leading and next-to-leading order Lagrangian in isospin limit $m_u = m_d$

$$\mathcal{L}_4 = \mathbb{L}_1 [\mathrm{Tr}\left(\partial^\mu \Sigma \partial_\mu \Sigma^\dagger\right)]^2 + \mathbb{L}_2 \mathrm{Tr}\left(\partial^\mu \Sigma \partial^\nu \Sigma^\dagger\right) \mathrm{Tr}\left(\partial_\mu \Sigma \partial_\nu \Sigma^\dagger\right)$$
$$+ \mathbb{L}_3 \frac{m_q \lambda}{f^2} \mathrm{Tr}\left(\partial^\mu \Sigma \partial_\mu \Sigma^\dagger\right) \mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right) + \mathbb{L}_4 \frac{(m_q \lambda)^2}{f^4} [\mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right)]^2$$

- Compute quark mass dependence of chiral condensate, pion mass, pion-pion scattering, ..., in terms of a few low-energy constants

$$\langle \overline{\psi} \psi \rangle = A_0 \left[1 + B_0\, m_q \left(\log m_q + C_0\right)\right] \qquad \checkmark$$

$$m_\pi^2 = A_1\, m_q \left[1 + B_1\, m_q \left(\log m_q + C_1\right)\right]$$

$$a_{\pi\pi}^{I=2} = A_2 \sqrt{m_q} \left[1 + B_2\, m_q \left(\log m_q + C_2\right)\right]$$

- Further applications: electroweak properties of pions require external fields

Incorporating External Fields in ChPT

- **Start by incorporating external gauge fields in QCD**

$$\mathcal{L}_\psi = \overline{\psi}_L i\not{D}_L \psi_L + \overline{\psi}_R i\not{D}_R \psi_R$$

E.g. external vector field $L_\mu = R_\mu = Qe\mathcal{A}_\mu$

$$(D_L)_\mu = \partial_\mu + ig\, G_\mu + iL_\mu$$
$$(D_R)_\mu = \partial_\mu + ig\, G_\mu + iR_\mu$$

Local invariance $[SU(2)_L] \otimes [SU(2)_R]$

$$\psi_L \longrightarrow L(x)\psi_L$$
$$\psi_R \longrightarrow R(x)\psi_R$$

and

$$L_\mu \longrightarrow L(x)L_\mu L^\dagger(x) + i[\partial_\mu L(x)]L^\dagger(x)$$
$$R_\mu \longrightarrow R(x)R_\mu R^\dagger(x) + i[\partial_\mu R(x)]R^\dagger(x)$$

- **Then incorporate external gauge fields in ChPT**

$\Sigma \to L(x)\Sigma R^\dagger(x)$ Need covariant derivative $D_\mu\Sigma \to L(x)[D_\mu\Sigma]R^\dagger(x)$

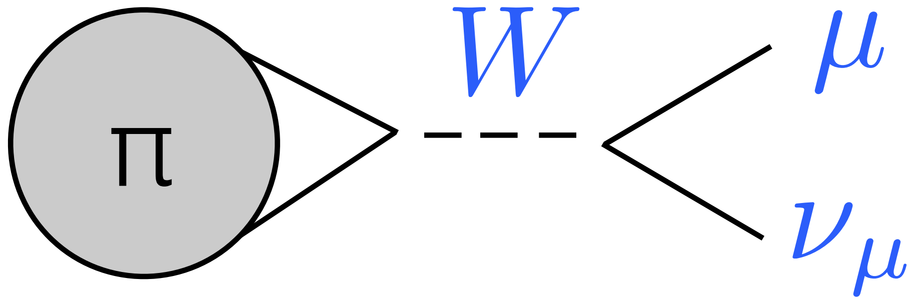
$$D_\mu\Sigma = \partial_\mu\Sigma + iL_\mu\Sigma - i\Sigma R_\mu^\dagger$$

Leading-order chiral Lagrangian [with external fields counted as $\mathcal{O}(p)$]

$$\mathcal{L}_2 = \frac{f^2}{8}\left[\mathrm{Tr}\left(D^\mu\Sigma D_\mu\Sigma^\dagger\right) + m_q\lambda\,\mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right)\right]$$

*Additional operators at higher orders

What is f?



$$\pi \to \mu + \nu_\mu$$

Pion weak decay $\Delta\mathcal{L} = W^{-\mu} J^+_{\mu L}$ $\qquad J^+_{\mu L} = \overline{u}_L \gamma_\mu d_L$

Strong part factorizes into QCD matrix element
(the rest you learned how to compute in QFT)

$$\langle\, 0\, |J^+_{\mu L}|\, \pi(\vec{p})\,\rangle = i p_\mu\, f_\pi$$

pion decay constant

$$\Gamma_{\pi\to\mu+\nu_\mu} = \frac{G_F^2}{8\pi} f_\pi^2 m_\mu^2 m_\pi |V_{ud}|^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2$$

$$f_\pi = 132\,\mathtt{MeV}$$

ChPT current matches the QCD current

$$\tau^+ = \frac{1}{2}(\tau^1 + i\tau^2)$$

$$J^a_{\mu L} = \left.\frac{\partial \mathcal{L}_\chi}{\partial L^{a\mu}}\right|_{L_\mu = 0} = \frac{f^2}{4}\mathrm{Tr}\left(i\tau^a \Sigma \partial_\mu \Sigma^\dagger\right) + \cdots = \frac{f}{2}\mathrm{Tr}\left(\tau^a \partial_\mu \phi\right) + \cdots$$

$$\langle\, 0\, |J^+_{\mu L}|\, \pi(\vec{p})\,\rangle = i p_\mu\, (f + \cdots)$$

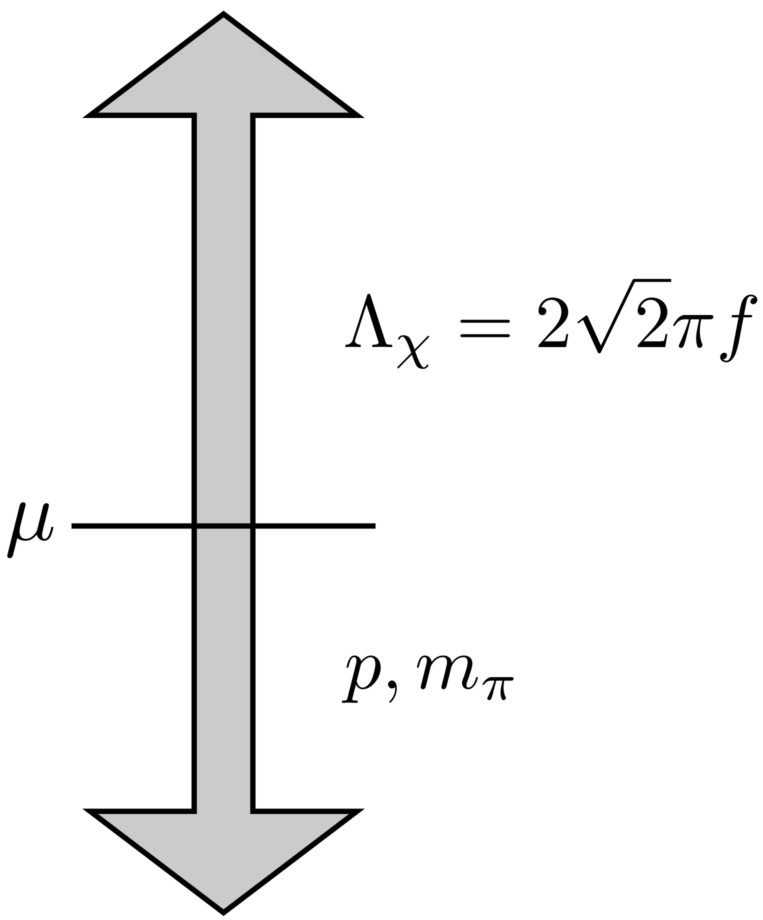
$$f_\pi = f\,[1 + B m_q\,(\log m_q + C)]$$

Dimensionless power counting $\qquad p^2/\Lambda_\chi^2$

$$\Lambda_\chi = 2\sqrt{2}\pi f \sim 1.2\,\mathtt{GeV} \qquad m_\pi^2/\Lambda_\chi^2$$

Exercise:

ChPT as an EFT



The masses of hadrons are modified by electromagnetism.

Construct all leading-order electromagnetic mass operators by promoting the electric charge matrix to fields transforming under the chiral group. (Notice that no photon fields will appear in the electromagnetic mass operators because there are no *external* photon lines.) Which pion masses are affected by the leading-order operators? Finally give an example of a next-to-leading order operator, or find them all.

Reminder: Asymptotic Expansions

- Non-analytic quark mass dependence implies asymptotic expansion (but obviously so: zero radius of convergence)

Toy Model $$F(x) = \int_0^\infty ds \frac{e^{-s}}{1+sx} \qquad 0 < x \ll 1$$

No series expansion about x=0

$$F(x) = \int_0^\infty ds\, e^{-s} \left(\sum_{j=0}^{\infty} (-s\, x)^j \right) \stackrel{!}{=} \sum_{j=0}^{\infty} (-x)^j \left(\int_0^\infty ds\, s^j\, e^{-s} \right)$$

Suggests approximation $$F_N(x) = \sum_{j=0}^{N} (-)^j j!\, x^j$$

$$|F(x) - F_N(x)| = x^{N+1} \int \frac{s^{N+1} e^{-s}\, ds}{1 + s\, x} \leq x^{N+1} (N+1)!$$

Large N $$\approx \sqrt{2\pi N} (xN)^N e^{-N} \sim \sqrt{\frac{2\pi}{x}} e^{-\frac{1}{x}}$$

Minimize

$$x \sim 1/N$$

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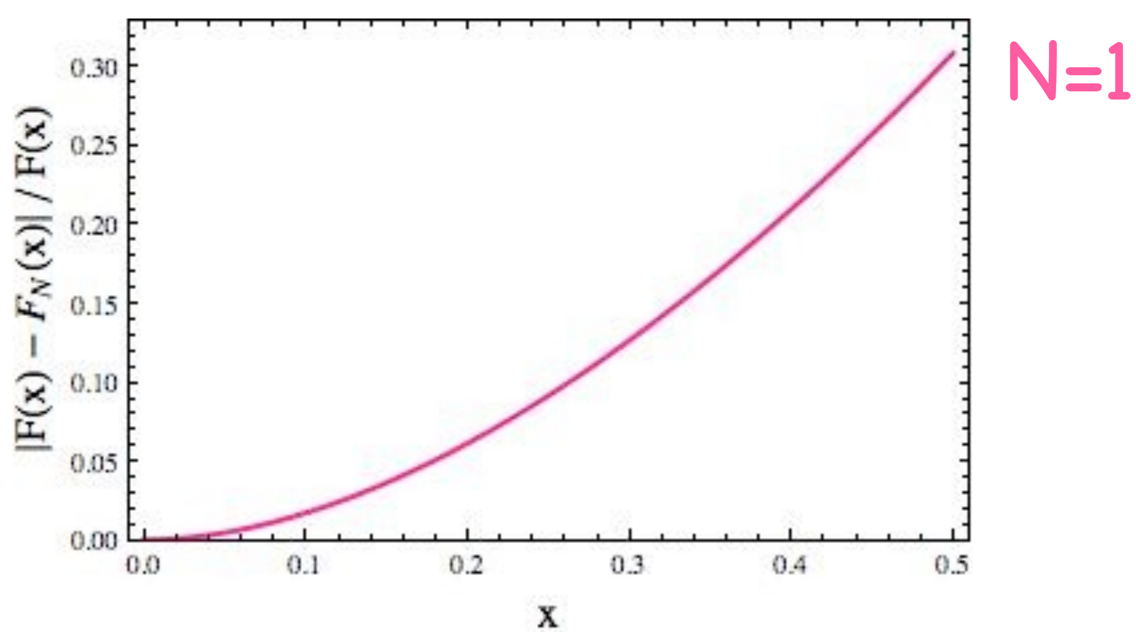

$$|F(x) - F_N(x)| \lesssim \sqrt{\frac{2\pi}{x}} e^{-\frac{1}{x}}$$

Minimize

$x \sim 1/N$

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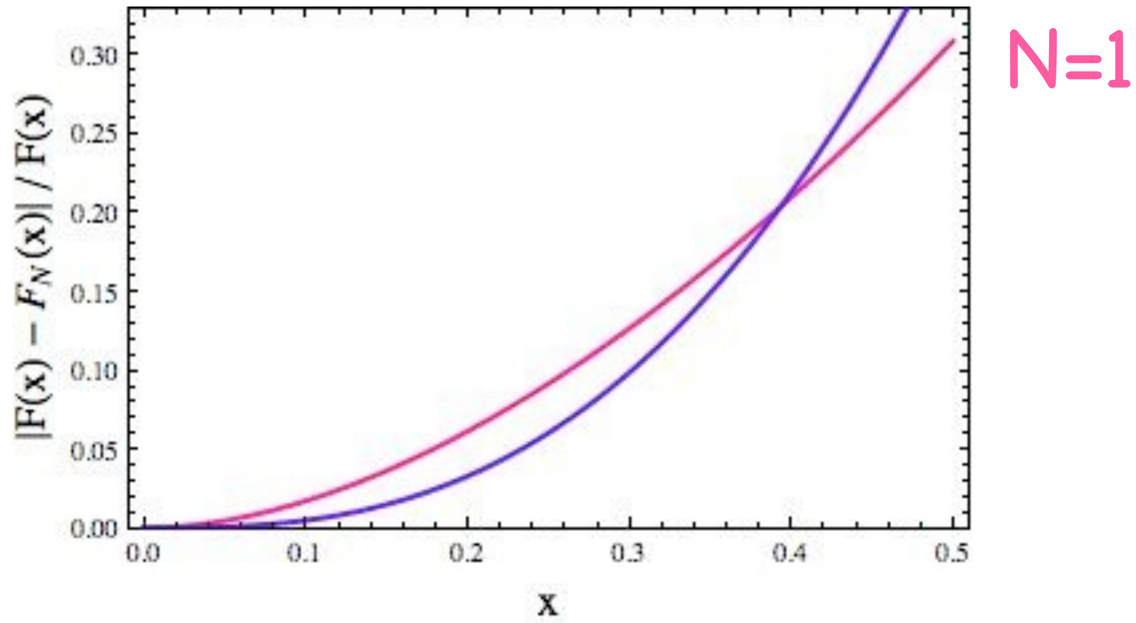
No series expansion about x=0

N=2



$$|F(x) - F_N(x)| \lesssim \sqrt{\frac{2\pi}{x}} e^{-\frac{1}{x}}$$

Minimize

$x \sim 1/N$

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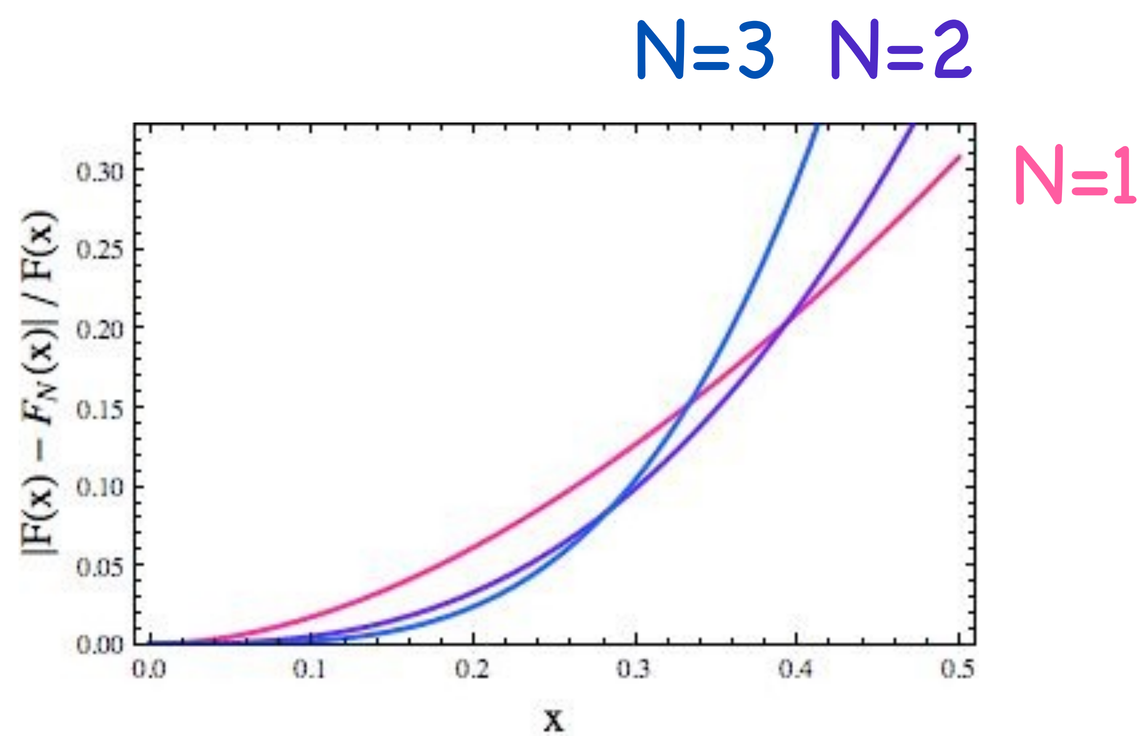

$$|F(x) - F_N(x)| \lesssim \sqrt{\frac{2\pi}{x}} e^{-\frac{1}{x}}$$

Minimize $x \sim 1/N$

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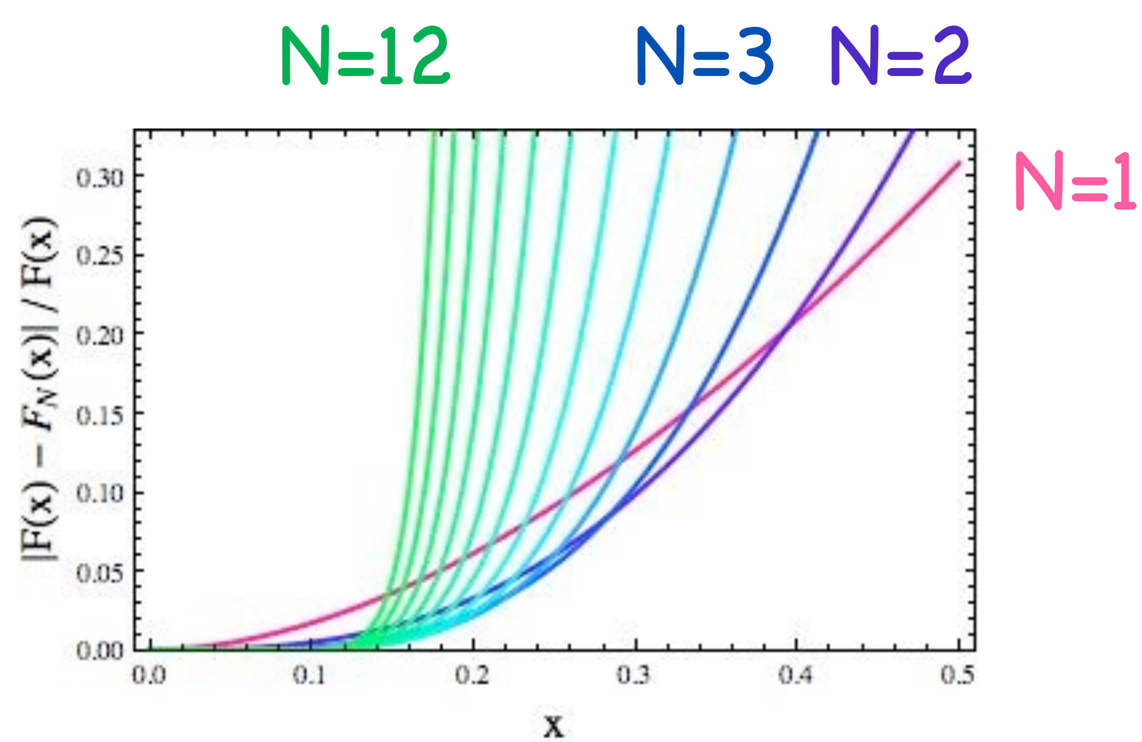

Include more terms: limits to smaller x

Make better at larger x by dropping terms

$$|F(x) - F_N(x)| \lesssim \sqrt{\frac{2\pi}{x}} e^{-\frac{1}{x}}$$

Minimize $x \sim 1/N$

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Chiral expansion

$$m_\pi^2/\Lambda_\chi^2 \sim 0.02$$
$$m_\pi^2/m_\rho^2 \sim 0.03$$
$$m_\pi^2/m_\sigma^2 \sim 0.08$$

(may even be OK for larger-than-physical pion masses)

EFT cannot capture non-analyticities from meson resonances

- Higher orders introduce more parameters (low-energy constants)
- Makes addressing convergence difficult without knowing the chiral limit values of these parameters

Three-Flavor Chiral Limit?

$$\mathcal{L}_\psi = \sum_{i=1}^{3} \overline{\psi}_i \left(i \not{D} - m_i \right) \psi_i$$

$$m_q / \Lambda_{\rm QCD} \sim 0.01$$

$$m_s / \Lambda_{\rm QCD} \sim 0.3$$

Ignore the warning signs

Three-Flavor Chiral Limit

Symmetries and their breaking

$$\mathcal{L}_\psi = \sum_{i=1}^{3} \overline{\psi}_i i \not{D} \psi_i + \ldots \qquad \langle \overline{\psi}\psi \rangle = \langle \overline{\psi}_R \psi_L \rangle + \langle \overline{\psi}_L \psi_R \rangle \neq 0$$

$$U(1)_V \otimes SU(3)_L \otimes SU(3)_R \longrightarrow U(1)_V \otimes SU(3)_V$$

$$\Sigma_{ij} \sim \langle \overline{\psi}_{jR} \psi_{iL} \rangle \qquad \Sigma_{ij}(x) = \delta_{ij} + \ldots \qquad SU(3)_L \otimes SU(3)_R / SU(3)_V$$

Goldstone modes (embedded similarly to before)

$$\Sigma = e^{2i\phi/f} \qquad \Sigma \to L\Sigma R^\dagger \qquad \Sigma \to V\Sigma V^\dagger \qquad \phi \to V\phi V^\dagger$$

$$\phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}$$

Three-Flavor Chiral Perturbation Theory

Explicit breaking $-\overline{\psi}_L\, m\, \psi_R - \overline{\psi}_R\, m\, \psi_L$ $m = \begin{pmatrix} m_q & & \\ & m_q & \\ & & m_s \end{pmatrix}$

$$U(1)_V \otimes SU(3)_L \otimes SU(3)_R \longrightarrow U(1)_V \otimes SU(3)_V$$

$\mathcal{O}(p^2)$ $$\mathcal{L}_\chi = \frac{f^2}{8} \mathrm{Tr}\left(\partial^\mu \Sigma \partial_\mu \Sigma^\dagger\right) + \lambda\, \mathrm{Tr}\left(m\Sigma + m\Sigma^\dagger\right)$$

Chiral perturbation theory (constructed similarly to before)

$\Sigma = e^{2i\phi/f}$ $\Sigma \to L\Sigma R^\dagger$ $\Sigma \to V\Sigma V^\dagger$ $\phi \to V\phi V^\dagger$

$\mathcal{O}(p^4)$ Seven Gasser-Leutwyler coefficients,
a few more when external fields are included

Exercises

In the strong isospin limit, there are two different quark masses but three meson masses of the pseudoscalar octet. Use the three-flavor chiral Lagrangian to derive

the constraint $\Delta_{\mathrm{GMO}} = \frac{4}{3}m_K^2 - m_\eta^2 - \frac{1}{3}m_\pi^2 = 0$, which was originally found by Gell-Mann and Okubo. What happens away from the strong isospin limit?

Revisit electromagnetic mass corrections in three-flavor chiral perturbation theory. Find all leading and next-to-leading order electromagnetic mass operators. Ignoring the up and down quark masses, which octet masses are affected by leading and next-to-leading order operators?

Accounting for strong and electromagnetic isospin breaking to leading order, determine the mass spectrum of the meson octet, and devise a way to compute the quark mass ratios, m_u/m_d m_d/m_s, using the experimentally measured masses.

Three-Flavor Chiral Perturbation Theory

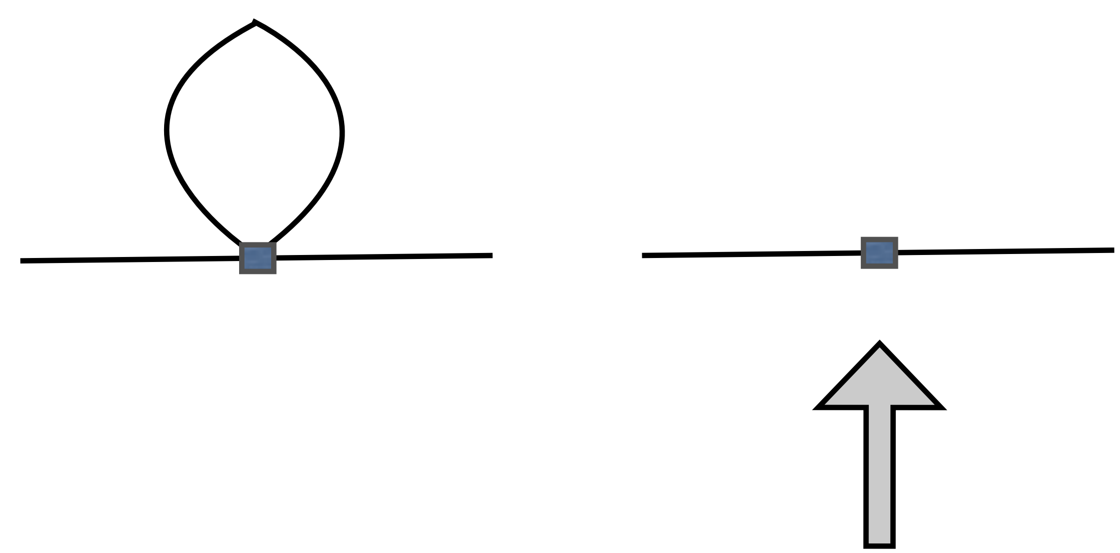
Gell-Mann Okubo mass relation $\Delta_{\mathrm{GMO}} = \frac{4}{3}m_K^2 - m_\eta^2 - \frac{1}{3}m_\pi^2 = 0$

$$m_{\pi^0} = 135.0\,\mathtt{MeV}$$
$$m_{K^0} = 497.6\,\mathtt{MeV}$$
$$m_\eta = 547.9\,\mathtt{MeV}$$

$$\Delta_{\mathrm{GMO}}/\overline{m}_\phi^2 \approx 15\%$$

Next-to-leading order corrections: $\mathcal{O}(p^2)$ one-loop + local terms from $\mathcal{O}(p^4)$

$$0 = \mathcal{O}(p^4) \sim \frac{m_\phi^4}{\overline{m}_\phi^2 \Lambda_\chi^2}$$



η most worrisome $\sim 35\%$

introduces free parameter

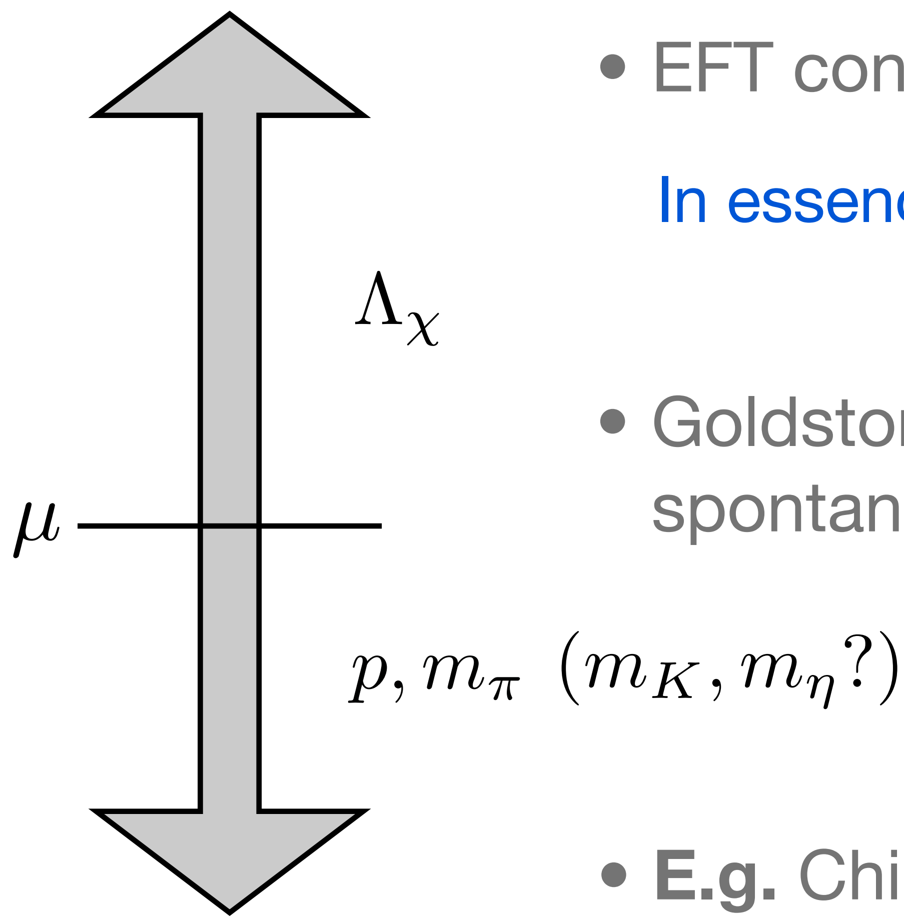
$$m_\pi^2/\Lambda_\chi^2 \sim 0.02$$
$$m_K^2/\Lambda_\chi^2 \sim 0.23$$
$$m_\eta^2/\Lambda_\chi^2 \sim 0.27$$

Pending numerical factors, $\mathcal{O}(p^6)$ contributions (which include two-loop diagrams) should be ~10%

Summary

III. Describing Goldstone bosons

- Spontaneous symmetry breaking can be systematically addressed in EFTs



- EFT construction is “bottom down”

  In essence effective d.o.f. arise non-perturbatively

- Goldstone boson dynamics consequence of pattern of spontaneous and explicit symmetry breaking

- **E.g.** Chiral perturbation theory provides the tool to account for light quark mass dependence of low-energy QCD observables.

  Size of light quark mass controls efficacy