Symmetry and Physical Law

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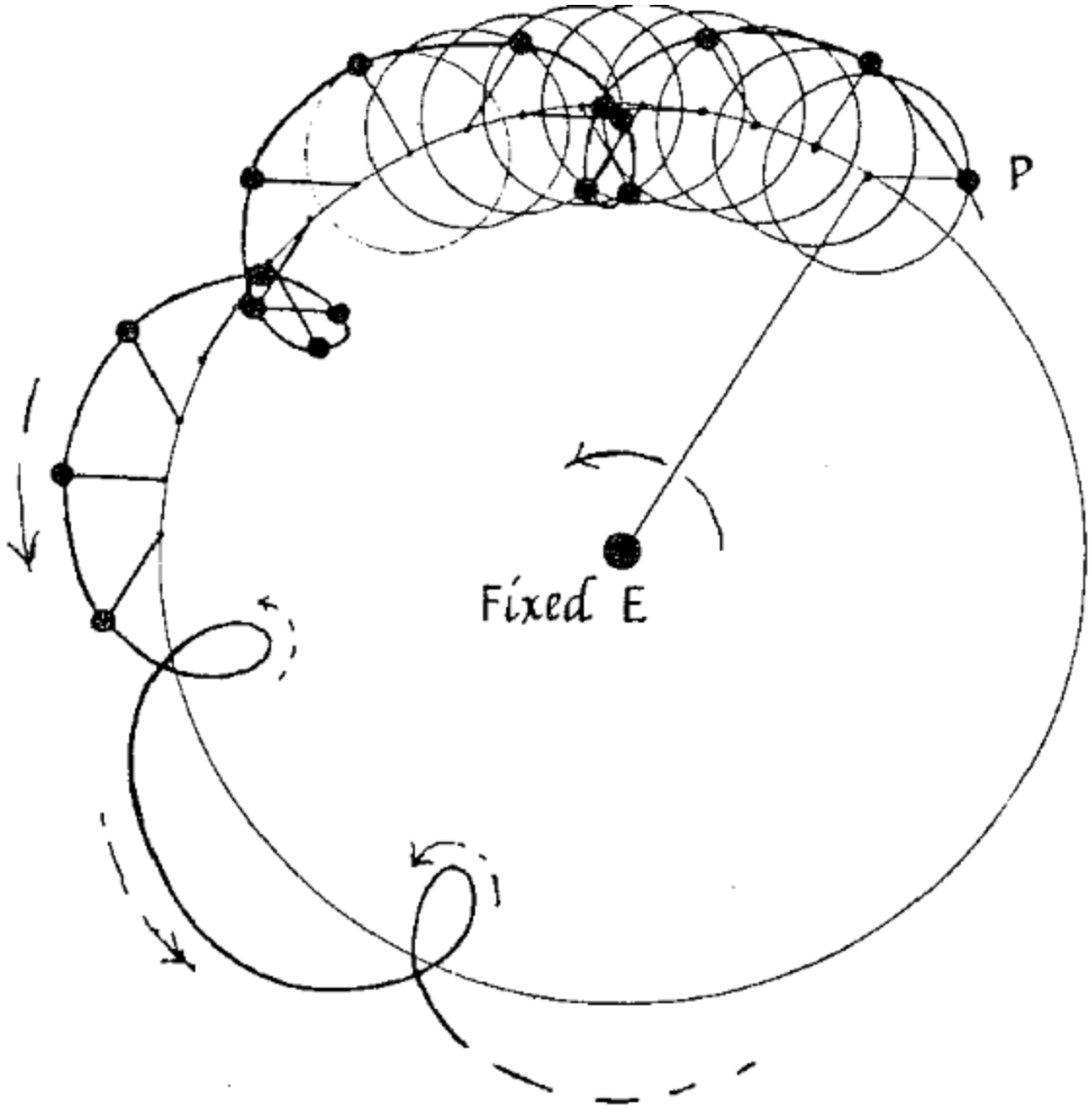
Mankind always fascinated by concept of symmetry

Pythagorans considered circle (sphere) to be most perfect two-dimensional (three-dimensional) objects because of symmetry

Stars were fixed in the heavenly spheres and planets moved in perfect circles

However, outer planets (Mars, Jupiter, Saturn) double back on themselves in path across the sky

Fixed by idea of epicycles (circles on circles)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Fixed E

Even Kepler used idea of symmetry in trying to understand planetary motion

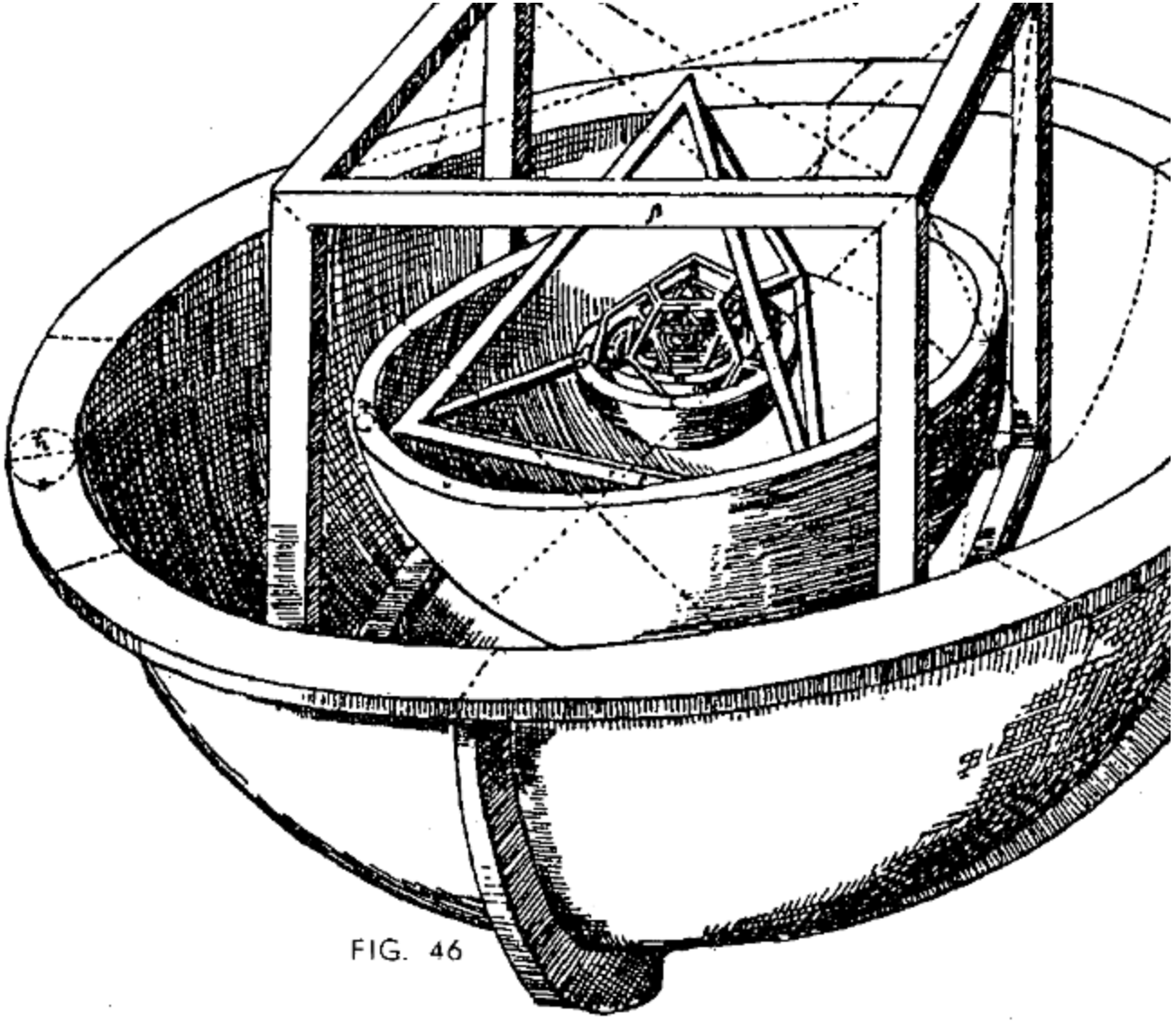
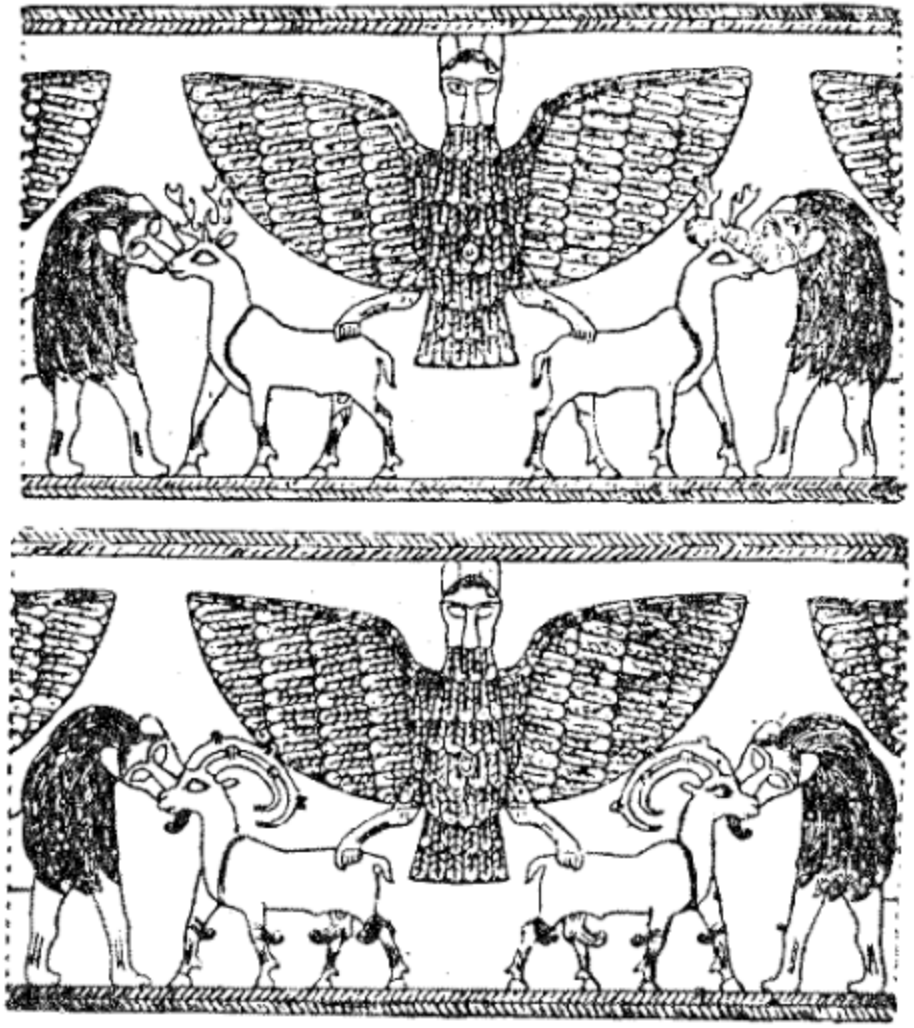


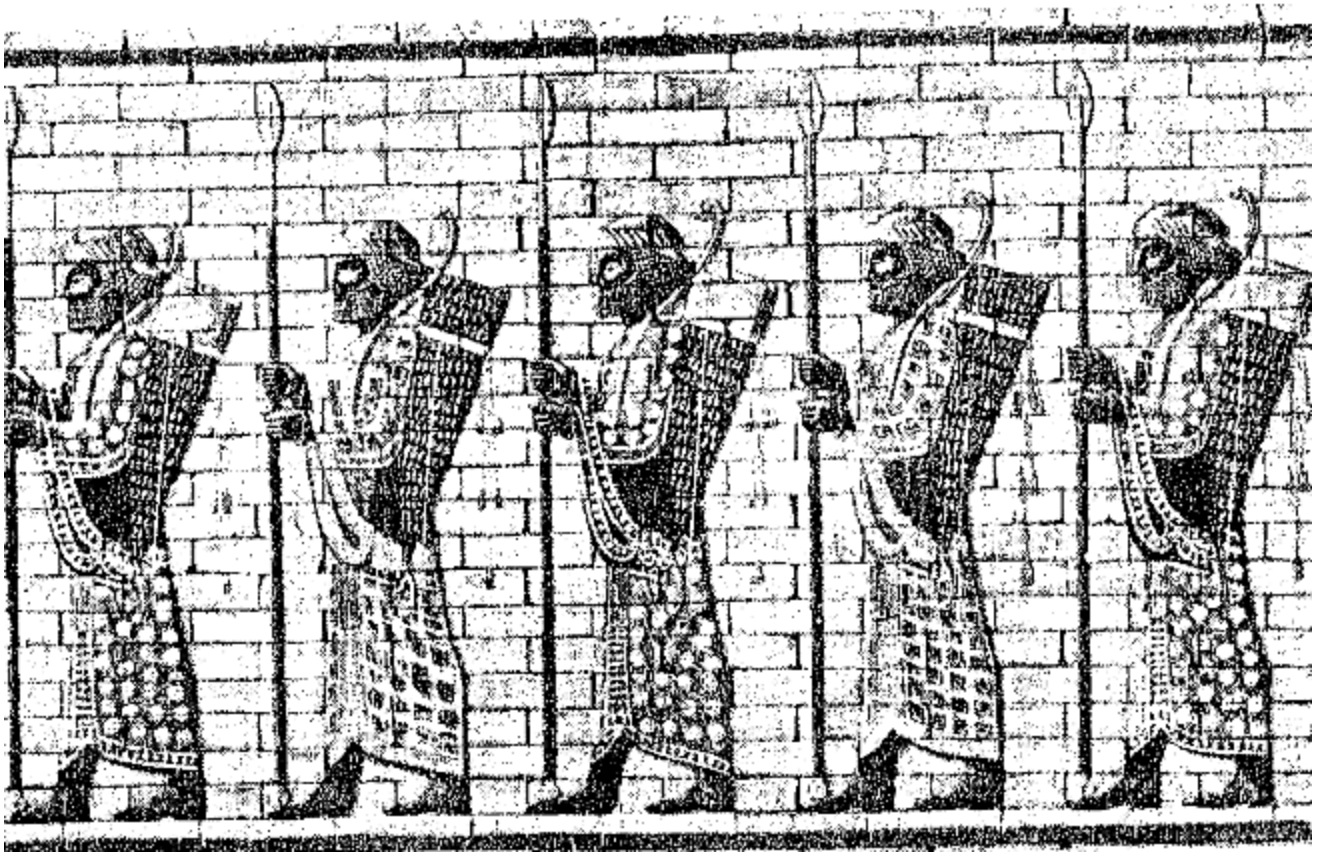
FIG. 46

Only with Newton, who pointed out it is laws of physics NOT orbits which are symmetry, did progress begin to be made in physics

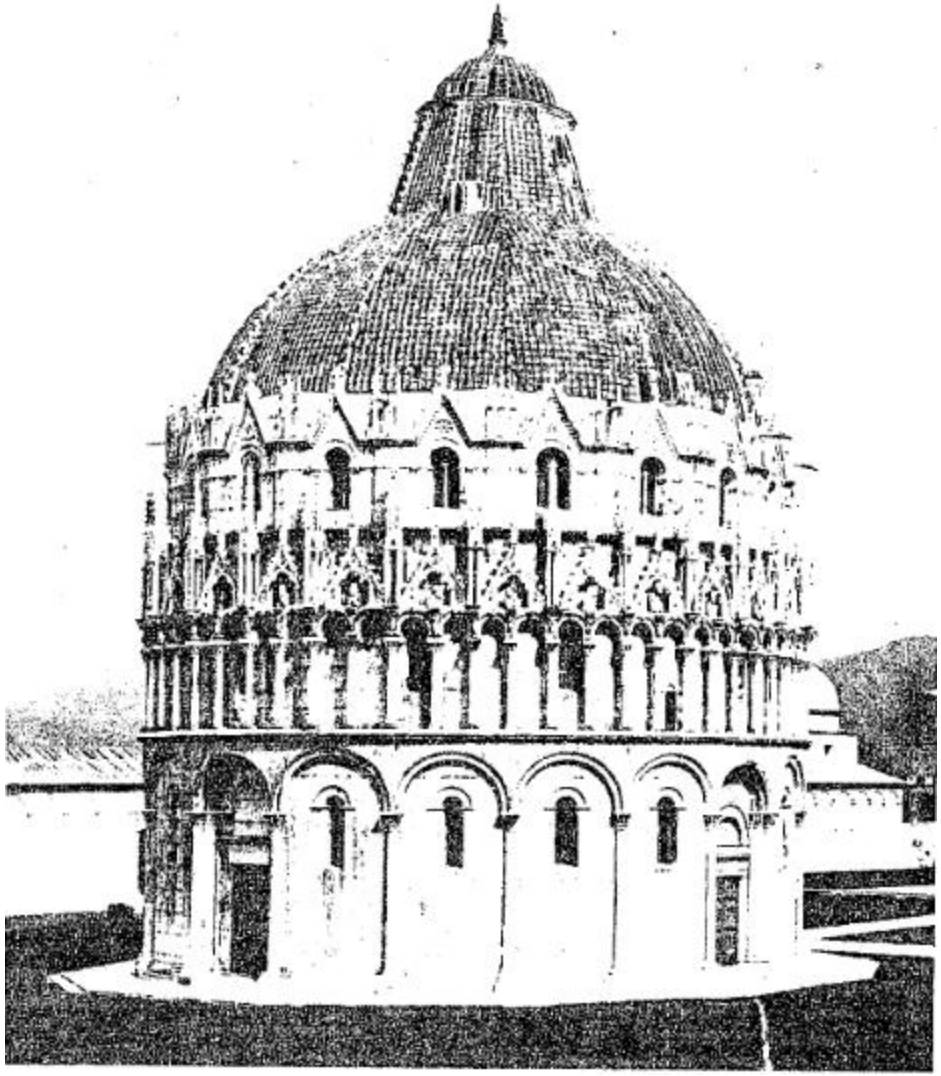
What is symmetry?



Bilateral Symmetry



Translational Symmetry



Rotational Symmetry

Hexagonal Symmetry

Symmetry and Symmetry Breaking

What is symmetry? Weyl's answer:

"Something is symmetric when it looks the same before and after you do something to it"

Importance to physics: Noether's theorem

Noether: Symmetry implies conservation law

[1]The mathematician Emmy Nöther was born in 1882 and did much of her work in Göttingen, where the great mathematicians Hilbert and Klein and the physicists Heisenberg and Schrödinger were professors. Fleeing the rise of Naziism, she spent her last years in the United States at Bryn Mawr and at the Institute for Advanced Study in Princeton. She died in 1935.

Examples:

a) $\mathcal{L}$ invariant under translation $\longrightarrow$ momentum conservation

b) $\mathcal{L}$ invariant under time translation $\longrightarrow$ energy conservation

c) $\mathcal{L}$ invariant under rotation $\longrightarrow$ angular momentum conservation

These are ONLY exact symmetries—all others are broken!

Classical Physics: Hamilton's Equations

$$\begin{aligned} \frac{d\vec{p}_i}{dt} &= -\frac{\partial H}{\partial \vec{r}_i} \\ \frac{d\vec{r}_i}{dt} &= \frac{\partial H}{\partial \vec{p}_i} \end{aligned}$$

Spatial Translation—Momentum Conservation

$$\delta H = \vec{\delta} \cdot \sum_{i=1}^{n} \frac{\partial H}{\partial \vec{r}_i} = -\vec{\delta} \cdot \frac{d}{dt} \sum_{i=1}^{n} \vec{p}_i = 0$$

Since $\vec{\delta}$ is arbitrary, we find

$$\sum_{i=1}^{n} \vec{p}_i = \text{const.}$$

Time Translation—Energy Conservation

$$\begin{aligned}\frac{dH}{dt} &= \sum_{i=1}^{n}\left[\frac{d\vec{r}_i}{dt}\cdot\frac{\partial H}{\partial \vec{r}_i}+\frac{d\vec{p}_i}{dt}\cdot\frac{\partial H}{\partial \vec{p}_i}\right]+\frac{\partial H}{\partial t}\\ &= \sum_{i=1}^{n}\frac{\partial H}{\partial \vec{p}_i}\cdot\frac{\partial H}{\partial \vec{r}_i}-\frac{\partial H}{\partial \vec{r}_i}\cdot\frac{\partial H}{\partial \vec{p}_i}=0\end{aligned}$$

That is, invariance under time translation implies that H=const.—conservtion of energy.

Rotation—Angular Momentum Conservation

Instead of Cartesian co-ordinates $\vec{r}_i$ and momentum $\vec{p}_i$, we use as generalized co-ordinates the angles θ, ϕ. The corresponding conjugate momenta are the angular momenta $\vec{L}_i$. Now suppose that we change the orientation of our apparatus by rotating the entire system in space through angle ϕ_0. If the Hamiltonian is invariant under this change (as we would certainly expect since there is no magic direction in space) then

$$\delta H = \phi_0 \sum_{i=1}^{n} \frac{\partial H}{\partial \phi} = -\phi_0 \sum_{i=1}^{n} \frac{d}{dt}(L_i)_z = 0$$

Since ϕ_0 is arbitrary, we find that

$$\sum_{i=1}^{n}(L_i)_z = \text{const.}$$

and similarly, we can show that all three components of the angular momentum are conserved. That is, invariance under spatial rotation imples that angular momentum is conserved.

Invariance and Quantum Mechanics

Observable q corresponds to hermitian operator $\hat{Q}$

$$q =< \psi|\hat{Q}|\psi >= \int d^3r \psi^*(\vec{r},t) Q \psi(\vec{r},t)$$

$$\begin{aligned}
\frac{dq}{dt} &= \int d^3r \left(\frac{\partial \psi^*}{\partial t} Q\psi + \psi^* \hat{Q} \frac{\partial \psi}{\partial t} \right) \\
&= i \int d^3r \left(\psi^* HQ\psi - \psi^* QH\psi \right) \\
&= i \int d^3r \psi^* [H,Q] \psi
\end{aligned}$$

In order that $dq/dt = 0$ for arbitrary wavefunctionψ we demand that

$$[\hat{H}, \hat{Q}] = 0$$

Classical Physics: Kepler Problem

If no use of symmetry have two coupled second order differential equations:

$$\frac{d^2\vec{r}}{dt^2} = -\frac{GM\vec{r}}{r^3}$$

with initial conditions

$$\vec{r}(0) = R_0\hat{e}_x, \quad \vec{v}(0) = V_0\hat{e}_y$$

Equivalent to *four* coupled first order equations.

Solve via iteration ($\epsilon = T/N_{max}$, $n = 1, 2, ..N_{max}$)

$$x[n] = x[n-1] + \epsilon * v_x[n-1]$$

$$y[n] = y[n-1] + \epsilon * v_y[n-1]$$

$$v_x[n] = v_x[n-1] - \epsilon * G * M * x[n-1] / (x[n-1]^2 + y[n-1]^2)^{\frac{5}{2}}$$

$$v_y[n] = v_y[n-1] - \epsilon * G * M * y[n-1] / (x[n-1]^2 + y[n-1]^2)^{\frac{5}{2}}$$

with initial conditions

$$x[0] = R_0;\ y[0] = 0;\ v_x[0] = 0;\ v_y[0] = V_0$$

Generates ellipse with sun at a focus.

Angular Momentum

Now use symmetry—for central force $d\vec{L}/dt = 0$. Then

$$E = \frac{L^2}{2mr^4}(\frac{dr}{d\phi})^2 + \frac{L^2}{2mr^2} - \frac{Gm}{r^2}$$

with

$$L = mr^2\frac{d\phi}{dt}$$

Solve directly with substitution $u = 1/r$

$$\phi - \phi_0 = \frac{L}{\sqrt{2m}}\int_{u_0}^{u}\frac{du'}{\sqrt{E + MmGu - \frac{L^2}{2m}u^2}}$$

yields

$$-\cos(\phi - \phi_0) = \frac{1 - \frac{L^2}{GMm^2}u}{\sqrt{1 + \frac{2EL^2}{m^3G^2M^2}}}$$

or

$$\frac{1}{r} = \frac{GMm^2}{L^2}(1 + e\cos(\phi - \phi_0))$$

with eccentricity ϵ given by

$$e = \sqrt{1 + \frac{2EL^2}{m^3G^2M^2}} < 1$$

Note that orbit is ellipse with sun at a focus. Also, NO precession of orbit. For Mercury observed precession is

$$\Delta\phi_{OBS} = 5600.73 \pm 0.41\text{"} \ /\text{century}$$

Influence of other planets (primarily Earth, Venus, Jupiter)

$$\Delta\phi_{planets} = 5557.62 \pm 0.20\text{"} \ /\text{century}$$

Discrepancy is

$$\Delta\phi_{OBS} - \Delta\phi_{planets} = 43.03\text{”} \ /\text{century}$$

and is due to general relativistic modification of Newtonian potential.

Runge-Lenz Vector

Use Runge-Lenz vector (first found by Laplace)

$$\vec{A} = \vec{v} \times \vec{L} - \frac{GMm}{r}\vec{r}$$

Can show that

$$\frac{d\vec{A}}{dt} = 0$$

Note that at perihelion or aphelion $\vec{v} \perp \vec{r}$ so $\vec{A}$ is along this direction. But $\vec{A}$ is independent of time so direction of perihelion or aphelion never changes—no precession!

For orbit evaluate scalar product

$$\vec{r} \cdot \vec{A} = \frac{1}{m}\vec{L}^2 - GMmr$$

so

$$r = \frac{\frac{1}{m}\vec{L}^2}{GMm + A\cos\theta}$$

so find ellipse with eccentricity $e = A/GMm$.

Note solution is algebraic—no differential equations to solve! Use of symmetry allows trivial solution.

HOW is symmetry broken?

Only *three* symmetry-breaking mechanisms in all of physics :

i) explicit symmetry breaking

ii) spontaneous symmetry breaking

iii) quantum mechanical symmetry breaking

We give examples of each!

i) Explicit Symmetry Breaking

Consider harmonic oscillator:

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2 x^2$$

Invariant under spatial inversion:

$$V_0(x) = -\frac{1}{2}m\omega_0^2 x^2 = -\frac{1}{2}m\omega^2(-x)^2 = V_0(-x)$$

Equilibrium configuration (ground state) determined via

$$\left.\frac{\partial V(x)}{\partial x}\right|_{x=x_E} = 0$$

$$i.e.,\ x_E = 0$$

consistent with symmetry!

Now break symmetry via constant force

$$L = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}m\omega_0^2 x^2 + \lambda x$$

Equilibrium now at $x_E = \frac{\lambda}{m\omega_0^2} \neq 0$!

Consistent with fact that symmetry is broken explicity—

$$\mathcal{L}(x) \neq \mathcal{L}(-x)$$

This is explicit symmetry breaking!

ii) Spontaneous Symmetry Breaking

Consider (frictionless) bead on hoop rotating in the earth's gravitational field about vertical axis—Lagrangian is

$$L = \tfrac{1}{2}m(R^2\dot{\theta}^2 + \omega^2 R^2 \sin^2\theta) + mgR\cos\theta)$$

Note symmetry condition

$$L(\theta) = L(-\theta)$$

Now find ground state:

$$\frac{\partial L}{\partial \theta} = m\omega^2 R^2 \sin\theta(\cos\theta - \tfrac{g}{\omega^2 R}) = 0$$

Two possibilities:

i) slow rotation—$\omega^2 < \frac{g}{R}$— $\cos\theta - \frac{g}{\omega^2 R} \neq 0$ and $\theta_E = 0$

ii) fast rotation—$\omega^2 > \frac{g}{R}$—equilibrium at $\theta_E = \pm \cos^{-1} \frac{g}{\omega^2 R}$

System is symmetric but ground state is not!

This is Spontaneous Symmetry Breaking!!

Quantum Mechanical Symmetry Breaking

Also called "anomalous" symmetry breaking. Occurs when symmetry is broken upon quantization. Hard to find non-field theory examples, but here is one:

Consider free particle, with Schrödinger equation:

$$-\frac{1}{2m}\nabla^2\psi = E\psi \equiv \frac{k^2}{2m}\psi$$

Partial wave solution is

$$\psi(\vec{r}) = \frac{1}{r}\chi_k(r)P_l(\cos\theta)$$

where

$$\left(-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+k^2\right)\chi_k(r)=0$$

Note invariance under scale transformation

$$r\to\lambda r \qquad k\to\frac{1}{\lambda}k$$

Consequence: For plane wave solution

$$e^{ikz}\overset{r\to\infty}{\longrightarrow}\frac{1}{2ikr}\sum_l(2l+1)P_l(\cos\theta)\left(e^{ikr}-e^{-i(kr-l\pi)}\right)$$

Note phase shifts $\ell\pi$ independent of energy–scale symmetry!

If a potential $V(\vec{r})$ then scale invariance is broken—

$$\psi^{(+)}(\vec{r}) \stackrel{r\to\infty}{\longrightarrow} \frac{1}{2ikr}\sum_l (2l+1)\,P_l(\cos\theta)(e^{i(kr+2\delta_l(k))}-e^{-i(kr-l\pi)})$$

Usually write as

$$\psi^{(+)}(\vec{r}) = e^{ikx} + \frac{e^{ikr}}{r} f_k(\theta)$$

with

$$f_k(\theta) = \sum_l (2l+1) \frac{e^{2i\delta_l(k)} - 1}{2ik} P_l(\cos\theta)$$

Phase shifts now a function of energy but OK, since scale symmetry broken.

Now look at *two* dimensions:

$$\psi^{(+)}(\vec{r}) \overset{r\to\infty}{\longrightarrow} e^{ikz} + \frac{1}{\sqrt{r}} e^{i(kr+\frac{\pi}{4})} f_k(\theta)$$

and

$$f_k(\theta) = -i \sum_{m=-\infty}^{\infty} \frac{e^{2i\delta_m(k)} - 1}{\sqrt{2\pi k}} e^{im\theta}$$

Now introduce potential

$$V(\vec{r}) = g\delta^2(\vec{r})$$

Note does NOT break scale symmetry.

Cross section is found to be

$$\frac{d\sigma}{d\Omega} \propto \frac{\pi}{2k} \frac{1}{(\ln \frac{k}{\mu}^2)}$$

or

$$\cot \delta_0(k) = \frac{1}{\pi} \ln \frac{k^2}{\mu^2} - \frac{2}{g}$$

Scale invariance broken upon quantization!

This is anomalous symmetry breaking!

Usual examples from quantum field theory. However, two examples from quantum mechanics—-two dimensional delta function potential and $1/r^2$ potential.

Physical example is electron in vicinity of electric dipole, so potential is

$$\phi(r) = \frac{p \cos\theta}{r^2}$$

which is separable. Write

$$\psi(r\theta, \phi) = u(r)\Theta(\theta)\Phi(\phi)/r$$

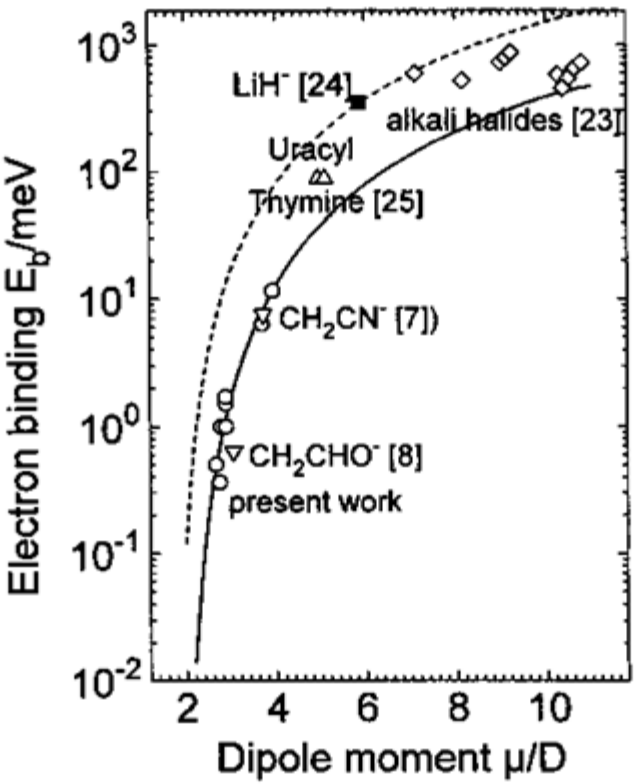
Then ϕ-independent solution satisfies

$$-\frac{d^2u}{dr^2}+\frac{\gamma}{r^2}u=-\kappa^2u$$
$$-\frac{d^2\Theta}{d\theta^2}-\mathrm{ctn}\theta\frac{d\Theta}{d\theta}+\lambda\cos\theta\Theta=\gamma\Theta$$

where $\lambda=2mep$ and $E=-\kappa^2/2m$. Satisfies scale invariance—$r\to\sigma r,\kappa\to\kappa/\sigma$. So if one bound state there exist bound state for all negative energies. Can regularize provided that $\gamma<-1/4$. Yields critical dipole moment

$$p_{crit} = 0.6393148771999813.../em$$

Problem first solved by Fermi and Teller and then rediscovered later. Connection with scale invariance recent–Camblong et al. For physical dipole—$p = Qd$—shown to be independent of d. Studied experimentally



Clear there exists critical value, but connection with anomaly unclear since $p_{crit}^{exp} \simeq 1.2 p_{crit}^{theory}$

In QFT anomaly found when Sutherland "proved" that decay $\pi^0 \to \gamma\gamma$ could not occur in chiral limit since $\partial \cdot V = \partial \cdot A = 0$. Shown by Adler, Jackiw and others that because of anomalous symmetry breaking

$$\partial \cdot V = 0 \quad \text{but} \quad \partial \cdot A = \frac{3\alpha}{8\pi} F^{\mu\nu} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$$

leading to

$$\Gamma_{\pi\gamma\gamma} = 7.74 \text{ eV}$$

in good agreement with particle data value.

Why not found earlier? Fermi and Teller calculation did not involve symmetry since they could do problem exactly, but could have.

Summary

Symmetry principles important to physics, but in general is broken

Three types of breaking:

i) Explicit Symmetry Breaking

ii) Spontaneous Symmetry Breaking

iii) Anomalous Symmetry Breaking

Symmetry and Effective Interactions

What is an effective interaction?

Answer: An interaction valid only in a limited range of parameter space

Classical Mechanics: Gravity

Gravitational Potential near the earth's surface:

i) $V(r) = -G\frac{mM_E}{r}$ valid for all $r > R_E$

ii) $V(r) \simeq mg(r - R_E)$ valid for $r - R_E << R_E$

Effective interaction form is ii) and is much easier to use!

Equivalently look at force:

i) $\vec{F} = -\frac{GmM_E}{r^2}\hat{r}$ valid for all $r > R_E$

ii) $\vec{F} = -m\vec{g}$ valid for $r - R_E << R_E$ Again second (effective) form is easier to use!

Why the Sky is Blue

A: Classical Physics Approach: Rayleigh Scattering

Atom is Harmonic Oscillator with frequency ω_0

Equation of motion in presence of ω photon

$$m\frac{d^2\vec{x}}{dt^2} = e\vec{E}_0 e^{-i\omega t} - m\omega_0^2\vec{x}$$

(Particular) solution

$$\vec{x}(t) = \frac{e\vec{E}_0}{m}\frac{e^{-i\omega t}}{\omega_0^2 - \omega^2}$$

has acceleration

$$\frac{d^2\vec{x}}{dt^2} = \frac{-\omega^2 e\vec{E}_0}{m}\frac{e^{-i\omega t}}{\omega_0^2 - \omega^2}$$

Accelerating charge radiates: then if $\omega^2 << \omega_0^2$

$$\text{Amp} \propto \omega^2 \longrightarrow \frac{d\sigma}{d\Omega} \sim \omega^4$$

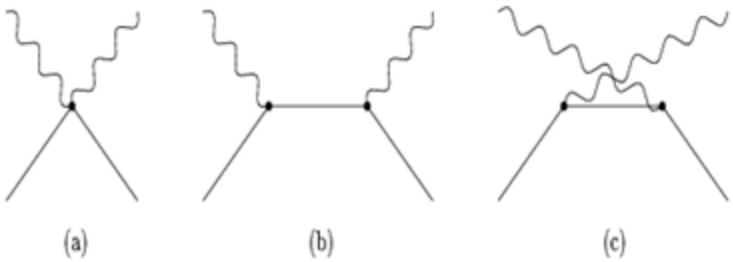
Blue light scatters >> Red light so sky is blue!

B: Quantum Mechanics Approach

Start with Hamiltonian

$$H = \frac{(\vec{p} - e\vec{A})^2}{2m} + e\phi$$

Feynman diagrams:



Leads to Kramers-Heisenberg amplitude

$$\text{Amp} = -\frac{e^2}{m\sqrt{2\omega_i\omega_f}}[\hat{\epsilon}_i \cdot \hat{\epsilon}_f$$

$$+\frac{1}{m}\sum_n (\frac{\hat{\epsilon}_f \cdot < 0|\vec{p}e^{-i\vec{q}_f\cdot\vec{r}}|n >< n|\vec{p}e^{i\vec{q}_i\cdot\vec{r}}|0 > \cdot\hat{\epsilon}_i}{\omega_i + E_0 - E_n}$$

$$+\frac{\hat{\epsilon}_i \cdot < 0|\vec{p}e^{i\vec{q}_i\cdot\vec{r}}|n >< n|\vec{p}e^{-i\vec{q}_f\cdot\vec{r}}|0 > \cdot\hat{\epsilon}_f}{E_0 - \omega_f - E_n})]$$

Homework problem gives, in limit when $\omega << E_n - E_0$

$$\frac{d\sigma}{d\Omega} = \lambda^2\omega^4|\epsilon_f^* \cdot \epsilon_i|^2 \left(1 + \mathcal{O}(\frac{\omega^2}{(E_n - E_0)^2})\right)$$

with

$$\lambda = \alpha_{em}\sum_n \frac{2|z_{n0}|^2}{E_n - E_0} \sim \alpha_{em}\frac{a_0^2}{m\alpha_{em^2}} \sim a_0^3$$

i.e.,

$$\frac{d\sigma}{d\Omega} \sim \omega^4 a_0^6$$

C: EFT Approach

Need effective Hamiltonian which is

i) quadratic in $\vec{A}$

ii) gauge-invariant

iii) rotational scalar

iv) P,T invariant

v) etc.

Simplest form is then

$$H_{eff} = -\frac{1}{2}4\pi\alpha_E\vec{E}^2 - \frac{1}{2}4\pi\beta_M\vec{H}^2$$

Here α_E, β_M have dimensions of volume—

$$\alpha_E\beta_M \sim a_0^3$$

Then

$$\frac{d\sigma}{d\Omega} \sim a_0^6\omega^4$$

Example from Quantum Mechanics

Consider system of atoms and photons, with path integral

$$Z = \int [d\psi][dA_\mu] \exp[i \int dt (L_{atom}(\psi) + L_\gamma(A_\mu) + L_{int}(\psi, A_\mu)]$$

If now integrate out atoms, find effective Lagrangian $L_{eff}[A_\mu]$

$$Z = \int [dA_\mu] \exp[i \int dt L_{eff}(A_\mu)]$$

where

$$\exp[i \int dt L_{eff}(A_\mu)] = \int [d\psi] \exp[i \int dt (L_{atom}(\psi) + L_\gamma(A_\mu) + L_{int}(\psi, A_\mu)]$$

Still have photons propagation but now with index of refraction.

If integrate out photons, find effective Lagrangian $L_{eff}[\psi]$

$$Z = \int [d\psi] \exp[i \int dt L_{eff}(\psi)]$$

where

$$\exp[i \int dt L_{eff}(\psi)] = \int [dA_\mu] \exp[i \int dt (L_{atom}(\psi) + L_\gamma(A_\mu) + L_{int}(\psi, A_\mu)]$$

Still have atoms but now with complex energies, corresponding to decay rates.

Condensed Matter Physics

Superconductivity

Underlying physics: electron-lattice interaction

Cooper pairing of anticorrelated electrons near Fermi surface leads to BCS Hamiltonian

$$H_{BCS} = \sum_{\vec{k},s} \psi^{\dagger}_{\vec{k},s} \left(i\frac{\partial}{\partial t} - \frac{\vec{k}^2}{2m} - \mu \right) \psi_{\vec{k},s}$$

$$-g \sum_{\vec{k}} \psi^{\dagger}_{\vec{k},\uparrow} \psi^{\dagger}_{-\vec{k},\downarrow} \psi_{-\vec{k},\downarrow} \psi_{\vec{k},\uparrow}$$

Can be solved with standard many-body methods—leads to energy gap and superconductivity provided temperature is low enough.

Physics is hidden, so use alternative approach: use finite temperature imaginary time methods to write Lagrangian in terms of two electron state ϕ.

Use identity

$$1 = \frac{\int [D\phi][D\phi^*] \exp -\kappa^2 \int_0^\beta d\tau \int d^3r (\phi - \frac{\sqrt{g}}{\kappa}\bar{\psi}^*_\alpha \bar{\psi}_\beta)(\phi - \frac{\sqrt{g}}{\kappa}\psi_\beta \psi_\alpha)}{\int [D\phi][D\phi^*] \exp -\kappa^2 \int_0^\beta d\tau \int d^3r \phi^* \phi}$$

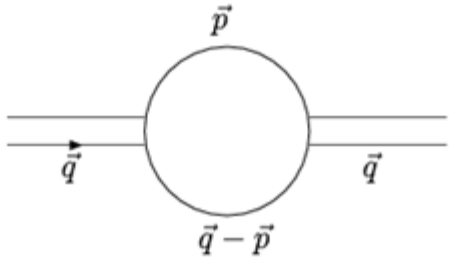
Then

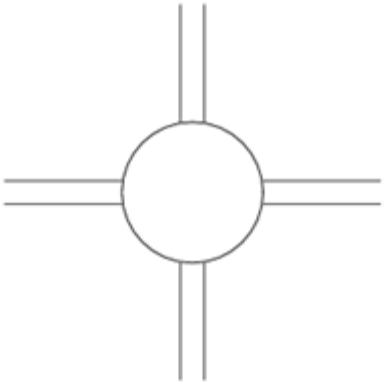
$$\begin{aligned} Z &= Const \times \int [D\psi][D\bar{\psi}][D\phi][D\phi^*] \\ &\times \exp -\sum_s \int_0^\beta d\tau \int d^3r \bar{\psi}_s (\frac{\partial}{\partial \tau} - \frac{(\vec{\nabla} + ie\vec{A})^2}{2m} - \mu)\psi_s \\ &\times \exp[-\kappa^2 \int_0^\beta d\tau \int d^3r \phi^* \phi + \sqrt{g}\kappa \int_0^\beta d\tau \int d^3r (\bar{\psi}_\alpha \bar{\psi}_\beta \phi + \psi_\beta \psi_\alpha \phi^*)] \end{aligned}$$

Now integrate out single electron fields to yield effective Lagrangian in terms of ϕ

$$Z = Const' \times \int [D\phi][D\phi^*] \exp -\kappa^2 \int_0^\beta d\tau \int d^3 r \phi^* L_{\text{eff}} \phi$$

Calculate diagrams:





Yields Landau-Ginzburg interaction

$$H_{eff} = c(T)\phi^\dagger \frac{(-i\vec{\nabla} - e^*\vec{A})^2}{2m^*}\phi + a(T)\phi^\dagger\phi + b(T)(\phi^\dagger\phi)^2 + \ldots$$

Here $c(T), b(T)$ monotonic but $a(T) \propto \log \frac{T}{T_c}$

Then if $T > T_c$ potential has shape of single well with minimum at $\phi^\dagger\phi \sim \rho = 0$—nonsuperconducting state.

If $T < T_c$ shape is double well with minimum at $\phi^\dagger\phi \sim \rho \neq 0$—superconducting state.

Physics clearly seen in Landau-Ginzburg form—spontaneously broken gauge invariance.

Quantum Chromodynamics

Want QCD analog of Superconductivity

Superconductor:

i) integrate out lattice from $\mathcal{L}(e^- - \text{lattice})$

ii) write Lagrangian in terms of $\phi \sim e^- e^-$

$\mathcal{L}(\phi)$ encodes basic physics—

Spontaneously broken gauge invariance

QCD:

i) integrate out gluons from $\mathcal{L}(q - -\text{gluons})$

ii) write Lagrangian in terms of $\phi \sim \bar{q}q$

$\mathcal{L}(\phi)$ encodes basic physics:

Spontaneously broken chiral invariance

Begin with QCD Lagrangian

$$\mathcal{L}_{QCD} = \bar{q}(i \not{D} - m)q - \frac{1}{4} G_{\mu\nu} G^{\mu\nu}$$

But

i) QCD written in terms of "wrong degree of freedom—q, A_μ

ii) nonlinear due to A^3, A^4 couplings

iii) strong coupling theory—$g^2/4\pi >> 1$

Solution is to use chirality operators:

$$q_{L,R} = \frac{1}{2}(1 \pm \gamma_5)q$$

Then

$$\bar{q}(i \not{D} - m)q = \bar{q}_L i \not{D} q_L + \bar{q}_R i \not{D} q_R - m(\bar{q}_L q_R + \bar{q}_R q_L)$$

Then $SU(3)_L \times SU(3)_R$ (chiral) symmetry in $m = 0$ limit

Axial symmetry spontaneously broken—Goldstone bosons

Define

$$U = \exp \frac{i}{F_\pi} \sum_{j=1}^{8} \lambda_j \phi_j$$

Then

$$\mathcal{L}_{eff} = \frac{F_\pi^2}{4}\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{F_\pi^2}{4}\mathrm{Tr}(B_0 m(U + U^\dagger)$$

Gives

i) Gell-Mann-Okubo relation—$3m_\eta^2 + m_\pi^2 - 4m_K^2 = 0$

ii) Weinberg scattering lengths—$a_0^0 = \frac{7m_\pi}{32\pi F_\pi^2}, a_0^2 = -\frac{m_\pi}{16\pi F_\pi^2}$

Tree level predictions. Loops give unitarity and also infinities. What to do? Weinberg (1979)—absorb divergences in phenomenological constants, just as in QED.

In higher orders gives chiral perturbation theory—Gasser and Leutwyler.

Not renormalizable—loops give new forms

$$\mathcal{L}_4 = \sum_{i=1}^{10} L_i \mathcal{O}_i = L_1 \Big[\mathrm{tr}(D_\mu U D^\mu U^\dagger) \Big]$$

$$+L_2 \mathrm{tr}(D_\mu U D_\nu U^\dagger) \cdot \mathrm{tr}(D^\mu U D^\nu U^\dagger)$$

$$+L_3 \mathrm{tr}(D_\mu U D^\mu U^\dagger D_\nu U D^\nu U^\dagger)$$

$$+L_4 \mathrm{tr}(D_\mu U D^\mu U^\dagger) \mathrm{tr}(\chi U^\dagger + U \chi^\dagger)$$

$$+L_5 \mathrm{tr}\left(D_\mu U D^\mu U^\dagger \left(\chi U^\dagger + U \chi^\dagger\right)\right) + L_6 \Big[\mathrm{tr}\left(\chi U^\dagger + U \chi^\dagger\right) \Big]^2$$

$$+L_7 \Big[\mathrm{tr}\left(\chi^\dagger U - U \chi^\dagger\right) \Big]^2$$

$$+L_8 \mathrm{tr}\left(\chi U^\dagger \chi U^\dagger + U \chi^\dagger U \chi^\dagger\right)$$

$$+iL_9 \mathrm{tr}\left(F^L_{\mu\nu} D^\mu U D^\nu U^\dagger + F^R_{\mu\nu} D^\mu U^\dagger D^\nu U\right)$$

$$+L_{10} \mathrm{tr}\left(F^L_{\mu\nu} U F^{R\mu\nu} U^\dagger\right)$$

where

$$D_\mu U = \partial_\mu U + \{A_\mu, U\} + [V_\mu, U]$$

and

$$F^{L,R}_{\mu\nu} = \partial_\mu F^{L,R}_\nu - \partial_\nu F^{L,R}_\mu - i[F^{L,R}_\mu, F^{L,R}_\nu]$$

$$F^{L,R}_\mu = V_\mu \pm A_\mu$$

. Renormalized parameters:

$$L^r_i = L_i - \frac{\gamma_i}{32\pi^2}\left[\frac{-2}{\epsilon} - \ln(4\pi) + \gamma - 1\right]$$

Found experimentally:

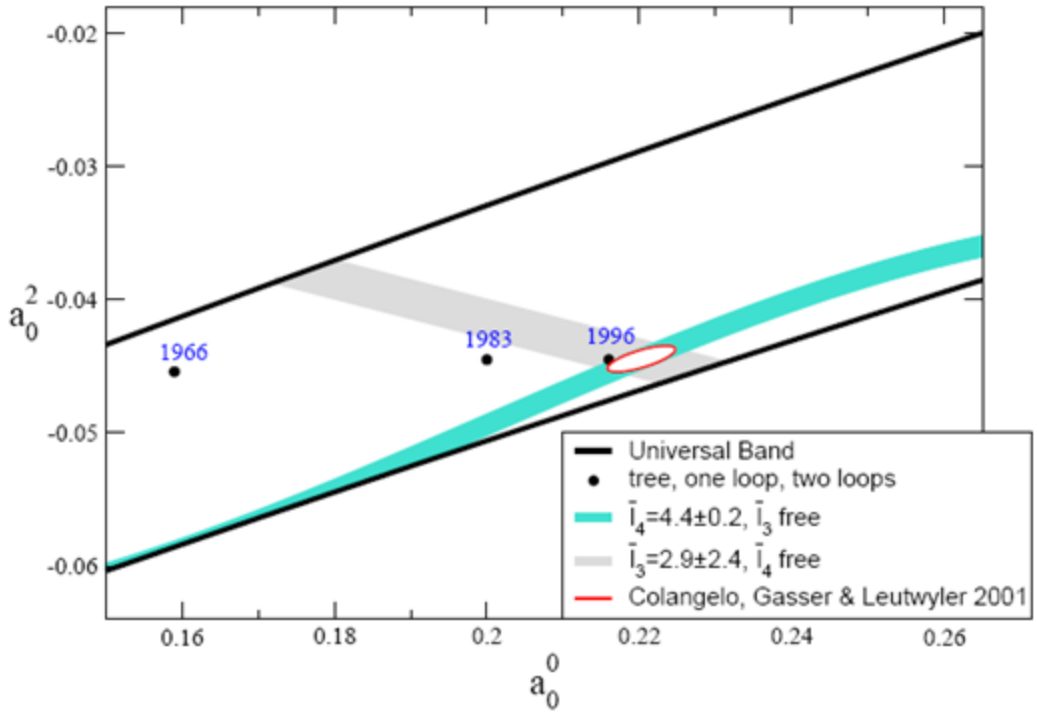
Coefficient	Value	Origin
L_1^r	0.65 ± 0.28	$\pi\pi$ scattering
L_2^r	1.89 ± 0.26	and
L_3^r	-3.06 ± 0.92	$K_{\ell 4}$ decay
L_5^r	2.3 ± 0.2	F_K/F_π
L_9^r	7.1 ± 0.3	π charge radius
L_{10}^r	-5.6 ± 0.3	$\pi \to e\nu\gamma$

Examples of predictions:

Reaction	Quantity	Theory	Experiment
$\pi^+ \to e^+\nu_e\gamma$	$h_V(m_\pi^{-1})$	0.027	0.029 ± 0.017
$\pi^+ \to e^+\nu_e e^+ e^-$	r_V/h_V	2.6	2.3 ± 0.6
$\gamma\pi^+ \to \gamma\pi^+$	$(\alpha_E + \beta_M)\,(10^{-4}\,\mathrm{fm}^3)$	0	1.4 ± 3.1
	$\alpha_E\,(10^{-4}\,\mathrm{fm}^3)$	2.8	6.8 ± 1.4
			12 ± 20
			2.1 ± 1.1

It works!

Chiral predictions for a_0^0 and a_0^2



Polarizabilities

What is a polarizability?

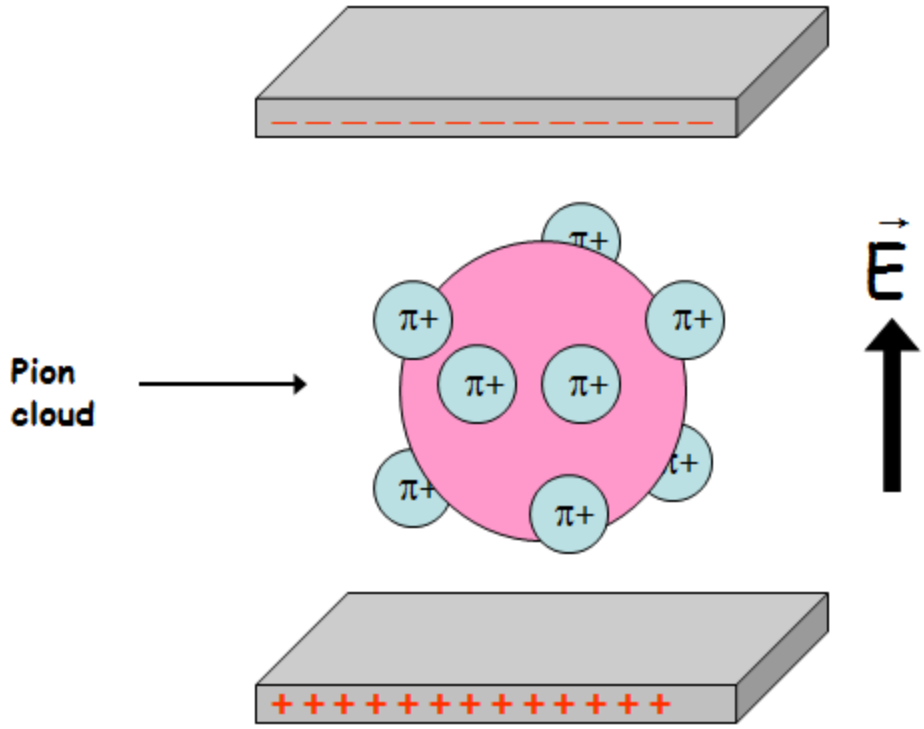
Answer: A measure of response of system to quasi-static electric and/or magnetic field. Simplest example: electric polarizability α_E—applied electric field $\vec{E}$ induces EDM $\vec{p}$

$$\vec{p} = 4\pi\alpha_E \vec{E}$$

Equivalently energy density is

$$u = -\frac{1}{2} 4\pi\alpha_E \vec{E}^2$$

Proton electric polarizability



Electric polarizability: proton between charged parallel plates

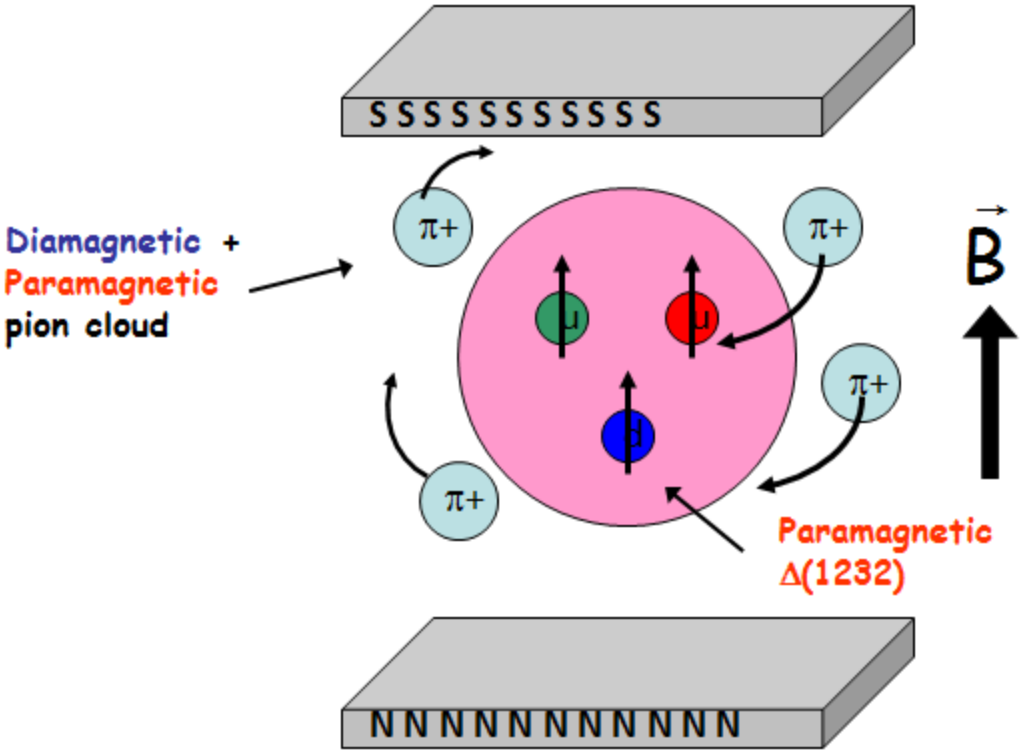
Similarly magnetic polarizability β_M—applied magnetic field $\vec{H}$ induces MDM $\vec{m}$

$$\vec{m} = 4\pi\beta_M \vec{H}$$

with energy density

$$u = -\frac{1}{2} 4\pi\beta_M \vec{H}^2$$

Proton magnetic polarizability



Magnetic polarizability: proton between poles of a magnetic

How to measure? Compton scattering—if

$$H = \frac{(\vec{p} - e\vec{A})^2}{2m} - \frac{1}{2}4\pi\alpha_E\vec{E}^2 - \frac{1}{2}4\pi\beta_M\vec{H}^2$$

with

$$\vec{E} = -\frac{\partial}{\partial t}\vec{A}, \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

then

$$T = \hat{\epsilon} \cdot \hat{\epsilon}' \left(-\frac{Q^2}{m} + \omega\omega' 4\pi\alpha_E \right) + \hat{\epsilon} \times \vec{k} \cdot \hat{\epsilon}' \times \vec{k}' 4\pi\beta_M$$

and

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2 \left(\frac{1}{2}(1+\cos^2\theta) - \frac{m\omega\omega'}{\alpha}\right.$$

$$\left. \cdot \left[\frac{\alpha_E + \beta_M}{2}(1+\cos\theta)^2 + \frac{\alpha_E - \beta_M}{2}(1-\cos\theta)^2\right] + \ldots\right)$$

Results from MAMI (TAPS), Illinois, Saskatoon for proton

$$\alpha_E^p = (12.0 \pm 0.6) \times 10^{-4}\,\mathrm{fm}^3, \quad \beta_M = (1.9 \mp 0.6) \times 10^{-4}\,\mathrm{fm}^3$$

What have we learned?

i) Size of α_E measures "stiffness" of system. For H atom well known calculation yields

$$\alpha_E^H = \frac{9}{2} a_B^3 = \frac{27}{8\pi} \mathrm{Vol}$$

while for proton

$$\alpha_E^p \simeq 3 \times 10^{-4} \mathrm{Vol}$$

so proton is very "stiff". Handwaving estimate is

$$\frac{\alpha_E^p / Vol.}{\alpha_E^H / Vol.} \sim \frac{E_{bind}^H / m}{E_{bind}^p / m} \sim \frac{\alpha_{em}^2}{\alpha_{strong}^2} \sim 10^{-4}$$

ii) Δ pole makes a very strong paramagnetic contribution to $\beta_M \sim +10 \times 10^{-4}\,\mathrm{fm}^3$ so proton must have strong diamagnetic component to cancel this.

iii) Presumably comes from meson cloud component—indeed simple quark model picture gives

$$\alpha_E^p = 2\alpha m_p < r_p^2 >^2 >> \alpha_E^{exp}$$

so meson contribution needed.

Estimate via chiral perturbation theory (Bernard, Kaiser, Meissner):

$$\alpha_E^p = \frac{\alpha g_A^2}{48\pi^2 F_\pi^2 M_n}\left[\frac{5\pi}{2\mu} + 18\log\mu + \frac{33}{2} + \mathcal{O}(\mu)\right] = 7.4$$

$$\beta_M^p = \frac{\alpha g_A^2}{48\pi^2 F_\pi^2 M_n}\left[\frac{\pi}{4\mu} + 18\log\mu + \frac{63}{2} + \mathcal{O}(\mu)\right] = -2.0$$

with $\mu = m_\pi/m_n$ in units of $10^{-4}\,\mathrm{fm}^3$.

If take only leading term ($\mathcal{O}(q^3)$ HBChipt), then

$$\alpha_E^p = 10\beta_M^p = \frac{5g_A^2}{96\pi F_\pi^2 m_\pi} = 12.2$$

in perfect agreement with experiment. Clearly accidental since $\mathcal{O}(q^4)$ calculation gives

$$\alpha_E^p = 10.5 \pm 2.0 \qquad \text{and} \qquad \beta_M^p = 3.5 \pm 3.6$$

Conclude:

i) symmetry a powerful constraint on physical systems

ii) use of symmetry makes solutions often MUCH simpler

iii) take time to find all possible symmetries of system—time well spent!