Introduction to High Energy Nuclear Collisions I (QCD at high gluon density)

Jamal Jalilian-Marian Baruch College, City University of New York

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Strong Interactions: Quantum ChromoDynamics (QCD)



Confinement (no free quarks/gluons)

Deeply Inelastic Scattering (DIS)







Deep-inelastic scattering (DIS) SLAC-MIT experiment of 1969

₹2



two unexpected results:





birth of the pre-QCD parton model

recall Rutherford experiments

Deeply Inelastic Scattering (DIS) probing hadron structure

Kinematic Invariants



Deeply Inelastic Scattering

first analysis of DIS does not require any knowledge about QCD



(can be easily generalized to W/Z-boson exchange)

with
$$L_{\mu\nu} = 2(k^{\mu}k'^{\nu} + k^{\nu}k'^{\mu} - g^{\mu\nu}k \cdot k')$$

Deeply Inelastic Scattering

Strong interactions: contained in the hadronic tensor ${f W}_{\mu
u}({f p},{f q})$

to all orders in the strong interaction $W_{\mu\nu}$ is given by the square of $\gamma^*(q) h(p) \rightarrow X$



symmetries (parity, Lorentz), hermiticity & current conservation tell us that $W^{\nu\mu}=W^{\mu\nu*}$ $q_{\mu}W^{\mu\nu}=0$

$$W_{\mu\nu}(p,q) = -\left(g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2}\right)F_1(x,Q^2) + \left(p_{\mu} - q_{\mu}\frac{p \cdot q}{q^2}\right)\left(p_{\nu} - q_{\nu}\frac{p \cdot q}{q^2}\right)\frac{1}{p \cdot q}F_2(x,Q^2)$$
structure functions



to obtain

DIS in the "naive" parton model

compare our result

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\mathrm{x}\mathrm{d}\mathrm{Q}^2} = \frac{4\pi\alpha^2}{\mathrm{Q}^4} [\mathbf{1} + (\mathbf{1} - \mathbf{y})^2] \frac{1}{2} \mathbf{e}_{\mathbf{q}}^2 \delta(\mathbf{x} - \xi)$$

to what one obtains with the hadronic tensor (parton level)

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4} \left[\left[1 + (1-y)^2 \right] F_1(x) + \frac{(1-y)}{x} (F_2(x) - 2xF_1(x)) \right]$$

and read off
$$F_2 = 2xF_1 = xe_q^2 \,\delta(x-\xi)$$
 Callan-Gross relation

ξp

protor

proton structure functions then obtained by weighting the parton str. fct. with the parton distribution functions (probability to find a parton with momentum ξ)

$$\begin{split} \mathbf{F_2} &= \mathbf{2xF_1} = \sum_{\mathbf{q},\mathbf{q}'} \int_0^1 d\xi \, \mathbf{q}(\xi) \, \mathbf{xe_q^2} \, \delta(\mathbf{x} - \xi) \\ &= \sum_{\mathbf{q},\mathbf{q}'} \mathbf{e_q^2} \, \mathbf{x} \, \mathbf{q}(\mathbf{x}) \\ \end{split} \end{split} \label{eq:F2} \end{split} \qquad \begin{array}{l} \text{DIS measures the sum of} \\ \text{quarks and anti-quarks} \end{array}$$

<u>space-time</u> picture of DIS

light cone variables

advantages: boosting is easy

separation of large and small components of vectors

$$P^{+} \equiv \frac{E + P_{z}}{\sqrt{2}}$$

$$P^{-} \equiv \frac{E - P_{z}}{\sqrt{2}} \quad (\mathbf{V}^{+}, \mathbf{V}^{-}, \mathbf{V}_{t}) \rightarrow (\mathbf{e}^{\omega} \mathbf{V}^{+}, \mathbf{e}^{-\omega} \mathbf{V}^{-}, \mathbf{V}_{t}) \text{ with } \mathbf{e}^{\omega} = \frac{\mathbf{Q}}{\mathbf{x} \mathbf{m}_{h}}$$

$$P_{t} = P_{t}$$

			_ //	*
4-vector	hadron rest frame	Breit frame		Р
$(p^+,p^-,ec{p}_T)$	$rac{1}{\sqrt{2}}(m_h,m_h,ec{0})$	$rac{1}{\sqrt{2}}(rac{Q}{x},rac{xm_h^2}{Q},ec{0})$		
$(q^+,q^-, ec{q_T})$	$\left egin{array}{c} rac{1}{\sqrt{2}}(-m_h x, rac{Q^2}{m_h x}, ec{0}) ight. ight.$	$rac{1}{\sqrt{2}}(-Q,Q,ec{0})$	q	
			. /	

<u>space-time</u> picture of DIS

t

 \mathcal{Z}

simple estimate for typical time-scale of interactions among the partons inside a fast-moving hadron:

rest frame: $\Delta x^+ \sim \Delta x^- \sim \frac{1}{2}$ х- ${m m}$ Breit frame: $\Delta x^+ \sim \frac{1}{m} \frac{Q}{m} = \frac{Q}{m^2}$ large $\Delta x^{-} \sim \frac{1}{m} \frac{m}{Q} = \frac{1}{Q}$ small world-lines interactions between of partons partons are spread out inside a fast moving hadron

How does this compare with the time-scale of the hard scattering?

space-time picture of DIS

now let the virtual photon meet our fast moving hadron ...



upshot:

- partons are free during the hard interaction
- hadron effectively consists of partons that have momenta $(p_i^+, p_i^-, \vec{p_i})$
- convenient to introduce momentum fractions $0 < \xi_i \equiv p_i^+/p^+ < 1$

What is inside a hadron: Parton model

Bjorken: \mathbf{Q}^2 , $\mathbf{S} \to \infty$ but keep $\mathbf{x}_{Bj} = \frac{\mathbf{Q}^2}{\mathbf{S}}$ fixed (scaling limit)

Structure functions depend only on x_{Bj}

Feynman:

Parton constituents of proton are "free" on time scale $1/Q << 1/\Lambda$ (interaction time scale between partons)



 X_{Bj} = fraction of hadron momentum carried by a parton = X_F

A hadron according to parton model



QCD - bound quarks

Collider experiment: Electron-Proton collisions at HERA (DESY, Hamburg, Germany)





early experiments (SLAC,...): scale invariance of hadron structure

Quantum ChromoDynamics (QCD)

Theory of strong interactions between quarks and gluons

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \sum_{i=1}^{\mathrm{f}} \bar{\Psi}^{\alpha}_i \left[i \not D - m_f \right]^{ij}_{\alpha\beta} \Psi^{\beta}_j$$

$$G^{a}_{\mu\nu}(x) \equiv \partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu} - gf^{abc}A^{b}_{\mu}A^{c}_{\nu}$$

 $a, b, c = 1, \cdots, 8$ color index: $\alpha, \beta = 1, 2, 3$ Lorentz index: $\mu, \nu = 0, 1, 2, 3$ spinor index: i, j = 1, 2, 3, 4

> Quarks: Fermions, spin 1/24x1 spinor, come in N_c colors 6 flavors (up, down,, top) carry electric charge

 $SU(N_c)$

 $N_c = 3$

Gluons: Bosons, spin 1 come in $N_c^2 - 1$ colors flavor blind have no electric charge

Gross, Wilczek; running of the coupling constant **Politzer ('73/'74)** Nobel prize 2004 asymptotic freedom value of strong coupling $\alpha_s = g^2/4\pi$ depends on distance r (i.e. energy Q) Abe "screening" of the charge "anti-screening" R α e e⁻ Ē e^{\dagger} Ŕ Ē R G R Ē R et e^{\dagger} Ē G в e⁻ (e⁻ Ē 4π $\alpha_s(Q^2) \approx$ $Q \sim 1/r$ who wins? $(\frac{11}{3}C_A - \frac{4}{3}T_F N_f) \ln(Q^2/\Lambda^2)$ typical hadronic scale O(200 MeV) Λ depends on N_f, pert. order and scheme

running of the coupling constant



expansion in powers of the coupling constant

Tests of QCD:

running of the coupling constant scaling violations in DIS vector boson production jet cross sections

.

QCD is firmly established we need to explore it and learn how to use it!

DIS in the QCD-improved parton model

we got a long way (parton model) without invoking QCD



now we have to study QCD dynamics in DIS

- this leads to similar problems already encountered in e⁺e⁻

let's try to compute the $O(\alpha_s)$ QCD corrections to the naive picture



 α_s corrections to the LO process

photon-gluon fusion

caveat: expect divergencies

related to soft/collinear emission or from loops

what to do with infinities? introduce "**regulator**" in the intermediate stages, remove it at the end

general structure of the $O(\alpha_s)$ corrections

using small quark/gluon masses we obtain:



convolute with the PDFs $\mathbf{F}_2(\mathbf{x}, \mathbf{Q}^2) \equiv \sum_{\mathbf{f}}^{f} \mathbf{e}_{\mathbf{f}}^2 \mathbf{x} [\mathbf{q}_{\mathbf{f}}(\mathbf{x}, \mathbf{Q}^2) + \bar{\mathbf{q}}_{\mathbf{f}}(\mathbf{x}, \mathbf{Q}^2)]$

DGLAP "evolution" equation:

scale dependence of parton distribution functions



best solved in Mellin moment space: set of ordinary differential eqs.; no closed solution in exp. form beyond LO (commutators of P matrices!)

properties of LO splitting functions



PDFs as bi-local operators

more physical formulation in Bjorken-x space:

matrix elements of bi-local operators on the light-cone

for quarks: (similar for gluons)



describing phenomena with transverse polarization ("Sivers function", ...)

interpretation as *<u>number operator</u>* only in "A+= o gauge"

DGLAP "evolution" equation: scale dependence of parton distribution functions <u>Dokshitzer-Gribov-Lipatov-Altarelli-Parisi</u>



DGLAP "evolution" equation: scale dependence of parton distribution functions





What drives the growth of parton distributions?

Splitting functions at leading order $O(\alpha_s^0)$ $(x \neq 1)$

$$\begin{split} P_{qq}^{(0)}(x) &= C_F \frac{1+x^2}{1-x} \\ P_{qg}^{(0)}(x) &= \frac{1}{2} \Big[x^2 + (1-x)^2 \Big] \\ P_{gq}^{(0)}(x) &= C_F \frac{1+(1-x)^2}{x} \\ P_{gg}^{(0)}(x) &= 2C_A \Big[\frac{x}{1-x} + \frac{1-x}{x} + x(1-x) \Big] \end{split}$$

At small x, only P_{gq} and P_{gg} are relevant.



\rightarrow Gluon dominant at small x!

The double log approximation (DLA) of

DGLAP is easily solved.

-- increase of gluon distribution at small x

 $\mathbf{xg}(\mathbf{x}, \mathbf{Q^2}) \sim \mathbf{e}^{\sqrt{lpha_{\mathbf{s}} \left(\mathbf{log1/x}
ight) \left(\mathbf{logQ^2}
ight)}}$

Keep this in mind: will be the <u>focus</u> of next part

pQCD at work:hadron-hadron collisions

What happens when two hadrons collide ?



concepts discussed so far fall in the category of collinear factorization in pQCD

Production of jets, hadrons, heavy quarks, ...



 $d\sigma = \sum_{i,j} \int dx_i dx_j f_i(x_i, \mu^2) f_j(x_j, \mu^2) d\hat{\sigma}_{ij}(\alpha_s(\mu_r), Q^2, \mu^2, x_i, x_j)$ non-perturbative linked hard scattering of but universal PDFs by μ two partons: pQCD

Large Hadron Collider (LHC)



pQCD: a success story



results now start to being used in global fits to constrain PDFs particularly sensitive to gluons

 $\mathbf{gg}
ightarrow \mathbf{gg} \quad \mathbf{gq}
ightarrow \mathbf{gq}$

two recent examples from the LHC:

1-jet and di-jet cross sections many other final-states available



 10^{3}

10⁻⁶

M_{ii} (GeV)

Precision QCD : P_{ij} @ **NNLO**

10000 diagrams, 10⁵ integrals, 10 man years, and several CPU years later:

 $F_{\rm pt}^{(2)}(x) \ = \ 16 C_d C_g n_f \Big(\frac{4}{3} (\frac{1}{x} + x^2) \Big[\frac{13}{3} {\rm H}_{-1,0} - \frac{14}{9} {\rm H}_0 + \frac{1}{2} {\rm H}_{-1} \zeta_2 - {\rm H}_{-1,-1,0} - 2 {\rm H}_{-1,0,0} \Big] \\$ $-H_{-1,2}\Big] + \frac{2}{3}(\frac{1}{x} - x^2)\Big[\frac{16}{3}\frac{\zeta_2}{\zeta_2} + H_{2,1} + 9\zeta_3 + \frac{9}{4}H_{1,0} - \frac{6761}{216} + \frac{571}{72}H_1 + \frac{10}{3}H_2 + H_1\zeta_2 - \frac{1}{6}H_{1,1} + \frac{10}{3}H_2 + H_1\zeta_2 - \frac{1}{6}H_{1,1} + \frac{10}{3}H_2 + \frac{10}$ $-3H_{1,0,0} + 2H_{1,1,0} + 2H_{1,1,1} + (1-x) \left[\frac{182}{9}H_1 + \frac{158}{3} + \frac{397}{36}H_{0,0} - \frac{13}{2}H_{-2,0} + 3H_{0,0,0,0} \right]$ $+\frac{13}{6}H_{1,0}+3aH_{1,0}+H_{-3,0}+H_{-3}\zeta_{2}+2H_{-2,-1,0}+3H_{-2,0,0}+\frac{1}{2}H_{0,0}\zeta_{2}+\frac{1}{2}H_{1}\zeta_{2}-\frac{9}{4}H_{1,0,0}$ $-\frac{3}{4}H_{1,1} + H_{1,1,0} + H_{1,1,1} + (1+x) \left[\frac{7}{12}H_0\zeta_2 + \frac{31}{6}\zeta_3 + \frac{91}{18}H_2 + \frac{71}{12}H_3 + \frac{113}{18}\zeta_2 - \frac{826}{27}H_0\right]$ $+\frac{5}{2}H_{2,0}+\frac{16}{3}H_{-1,0}+6xH_{-1,0}+\frac{31}{6}H_{0,0,0}-\frac{17}{6}H_{2,1}+\frac{117}{20}J_2^{-2}+9H_0J_3+\frac{5}{2}H_{-1}J_2+2H_{2,1,0}$ $+\frac{1}{2}H_{-1,0,0}-2H_{-1,2}+H_2\zeta_2-\frac{7}{2}H_{2,0,0}+H_{-1,-1,0}+2H_{2,1,1}+H_{3,1}-\frac{1}{2}H_4\Big]+5H_{-2,0}+H_{2,1}$ $+H_{6,0,0,0}-\frac{1}{2}\zeta_{2}^{'}z^{2}+4H_{-3,0}+4H_{6}\zeta_{3}-\frac{32}{9}H_{3,0}-\frac{29}{12}H_{9}-\frac{235}{12}\zeta_{2}-\frac{511}{12}-\frac{97}{12}H_{1}+\frac{33}{4}H_{2}-H_{3}$ $-\frac{11}{2}H_0\zeta_2-\frac{11}{2}\zeta_3-\frac{3}{2}H_{3,0}-10H_{0,0,0}+\frac{2}{3}s^2\Big[\frac{83}{4}H_{0,0}-\frac{243}{4}H_0+10\zeta_3+\frac{511}{8}+\frac{97}{8}H_1-\frac{4}{3}H_3\Big]$ $-4\zeta_{3}-H_{0}\zeta_{2}+H_{3}+H_{3,0}-6H_{-3,0}\Big]\Big)+16C_{F}n_{f}^{-2}\Big(\frac{2}{27}H_{0}-2-H_{2}+\zeta_{2}+\frac{2}{3}n^{2}\Big[H_{2}-\zeta_{2}+3$ $-\frac{19}{6}H_0$ + $\frac{2}{6}(\frac{1}{2}-x^2)$ [H_{1,1} + $\frac{5}{3}H_1$ + $\frac{2}{3}$] + (1-x) [$\frac{1}{6}H_{1,1}$ - $\frac{7}{6}H_1$ + xH_1 + $\frac{35}{27}H_0$ + $\frac{185}{54}$] $+\frac{1}{3}(1+x)\left[\frac{4}{3}H_2-\frac{4}{3}\zeta_2+\zeta_3+H_{2,1}-2H_3+2H_0\zeta_2+\frac{29}{6}H_{6,6}+H_{6,6,9}\right]\right)+16C_p^{-2}n_f\left(\frac{85}{12}H_1\right)$ $-\frac{25}{4}H_{0,0}-H_{0,0,0}+\frac{583}{12}H_0-\frac{101}{54}+\frac{73}{4}J_2-\frac{73}{4}H_2+H_3-5H_{3,0}-H_{2,1}-H_0J_3+x^2\Big[\frac{55}{12}$ $\frac{85}{12}H_1 - \frac{22}{3}H_{3,0} - \frac{109}{6} - \frac{13}{54}H_0 + \frac{28}{9}t_{3}' - \frac{23}{9}H_2 - \frac{16}{3}H_0t_2' + \frac{16}{3}H_3 + 4H_{2,0} + \frac{4}{3}H_{2,1} - \frac{26}{3}\zeta_3$ $+\frac{22}{3}H_{0,0,0}\Big]+\frac{4}{3}(\frac{1}{x}-x^2)\Big[\frac{23}{12}H_{1,0}-\frac{523}{144}H_1-3\zeta_3+\frac{55}{16}+\frac{1}{2}H_{1,0,0}+H_{1,1}-H_{1,1,0}-H_{1,1,1}\Big]$ $+(1-x)\left[\frac{1}{2}H_{1,0,0}+\frac{7}{12}H_{1,1}-\frac{2743}{72}H_{0}-\frac{53}{12}H_{0,0}-\frac{251}{12}H_{1}-\frac{5}{4}L_{2}+\frac{5}{4}H_{2}-\frac{8}{3}H_{1,0}+3tH_{1,0}\right]$ $+3H_{6}\zeta_{2}-3H_{3}-H_{1,1,0}-H_{1,1,1}\Big]+(1+x)\Big[\frac{1669}{216}+\frac{5}{2}H_{0,0,0}+4H_{2,1}+7H_{2,0}+10\pi\zeta_{3}-\frac{37}{10}\zeta_{2}\Big]$ $-7H_0\zeta_3 + 6H_{0,0}\zeta_3 - 4H_{0,0,0} + H_{2,0,0} - 2H_{2,1,0} - 2H_{3,1,1} - 4H_{3,0} - H_{3,1} - 6H_4$

$$\begin{split} P_{61}^{(0)}(s) &= 16C_{4}C_{5}P_{4}\left(\rho_{40}(s)\right)\frac{32}{2}H_{5}C_{5}-44t_{1,1,1}+3H_{2,0,0}-\frac{13}{2}H_{1,2}+\frac{2}{2}H_{1,1,0}+3H_{2,1,0}\\ +H_{6}C_{5}-2H_{2,1,1}+4H_{5}C_{5}-\frac{17}{12}H_{5}C_{5}-\frac{5}{2}H_{0,0}+\frac{4}{2}H_{5}-\frac{4}{2}H_{5}-\frac{3}{2}H_{1,0,0}-\frac{1}{2}H_{1,0,0}\\ -\frac{385}{22}H_{1,0}-\frac{31}{2}H_{1,1}-\frac{113}{2}H_{1}+\frac{2}{2}H_{5}+\frac{5}{2}H_{5}(s)+\frac{2}{9}H_{0,0}+\frac{173}{2}H_{1}-\frac{229}{2}+\frac{223}{216}H_{0}\\ +H_{6}(s)+3H_{1,1,2}+9H_{1,0}(s)+6H_{1,0}(s)+3H_{1,1,0}-4H_{1,1,1}-3H_{1,1,2}-6H_{1,1}-6H_{1,1}) \end{split}$$

 $-2H_{3,0} - \frac{13}{2}H_{6}\zeta_{2} - 13H_{-3,0} - \frac{13}{2}H_{3,1} + \frac{15}{2}H_{3} - \frac{2005}{64} + \frac{157}{4}\zeta_{2} + 8\zeta_{3} + \frac{1291}{432}H_{1} + \frac{55}{12}H_{1,1}$ $+\frac{3}{2}H_2+\frac{1}{2}H_{2,1}+\frac{27}{4}H_{-1,0}-\frac{11}{2}H_{1,0,0}-8H_{2,0,0}-4\zeta_2^{-2}+\frac{3}{2}H_{1,2}-H_{2,2}+\frac{5}{2}H_1\zeta_2+8H_{-1,-1,0}$ $+4H_{2,0}+\frac{3}{2}H_{2,1,1}-H_{-1}\zeta_{2}+7H_{2}\zeta_{2}+6H_{-3}\zeta_{2}+12H_{-2,-1,0}-6H_{-3,0,0}+x\Big[3H_{1,1,1}-H_{0,0}\zeta_{2}+2H_{-2,-1,0}-6H_{-3,0,0}+x\Big]$ $+\frac{9}{2}H_{-1,0,0}-\frac{35}{8}H_{1,0}+2H_4+3H_{1,1,0}+H_{-1,2}\right)+16C_4^{-2}C_p\left(x^2\left|\frac{2}{3}H_1\zeta_2-\frac{2105}{81}-\frac{77}{18}H_{0,0}\right.\right)$ $-6H_3+\frac{16}{2}\zeta_3-10H_{-1,0}-\frac{14}{2}H_{2,0}-\frac{2}{3}H_{-1}\zeta_3-\frac{14}{3}H_{0,0,0}+\frac{104}{9}H_2-\frac{4}{3}H_{1,0,0}+\frac{37}{9}H_{1,1}$ $+\frac{4}{3}H_{-1,-1,0}-\frac{104}{9}J_{22}'-\frac{8}{3}H_{2,1}+\frac{145}{18}H_{1,0}+\frac{4}{3}H_{-1,2}+\frac{2}{3}H_{1,1,1}-\frac{109}{27}H_{1}+\frac{8}{3}H_{-1,0,0}+6H_{0}J_{22}'$ $+4H_{-2,0}+\frac{584}{27}H_{0}\Big]+p_{60}(x)\Big[\frac{7}{2}H_{1}\zeta_{3}+\frac{138305}{2592}-\frac{1}{3}H_{2,0}+\frac{13}{4}H_{-1}\zeta_{2}+2H_{2,1,1}+\frac{11}{2}H_{1,0,0}$ $+4H_{3,l}-\frac{43}{6}H_{1,l,1}-\frac{109}{12}\zeta_{2}-\frac{17}{3}H_{2,l}-\frac{71}{24}H_{1,0}-\frac{11}{6}H_{-2,0}-\frac{21}{2}\zeta_{3}+\frac{3}{2}H_{1,0,0,0}-H_{1,-2,0}$ $+\frac{395}{54}H_{0}-2H_{1,0}\zeta_{2}-H_{1,1}\zeta_{2}-\frac{55}{12}H_{1,1,0}+2H_{1,1,0,0}+4H_{1,1,1,0}+2H_{1,1,1,1}+4H_{1,1,2}-\frac{55}{12}H_{1,2,1,1}+2H_{1,1,1,1}+4H_{1,1,2}-\frac{55}{12}H_{1,2,1,1}+2H_{1,1,1,1}+2H_{1,$ $+6H_{1,2,0} + 4H_{1,2,1} + 4H_{1,3} + 3H_{2,1,0} + 3H_{2,2} + p_{gq}(-x) \left[\frac{23}{2}H_{-1}\zeta_3 + 5H_{-2}\zeta_2 + 2H_{-2,-1,0}\right]$ $+\frac{109}{15}H_{-1,0}+H_0\zeta_3+\frac{17}{5}\zeta_3^{-2}+\frac{1}{6}H_1\zeta_2+2H_2\zeta_2-\frac{65}{24}H_{1,1}-\frac{19}{2}H_{-1,-1,0}-4H_{3,0}-3H_{2,0,0}$ $\begin{array}{c} -7H_{-2,0,0} = \frac{3}{2}H_{-1,2} + \frac{3379}{216}H_1 - 4H_{-2,2} - \frac{49}{6}H_{-1,0,0} - \frac{11}{2}H_{-1,0,0,0} - 13H_{-1,-1}\zeta_2 - 8H_{-1,3} \\ -6H_{-1,-1,-1,0} + 12H_{-1,-1,0,0} + 10H_{-1,-1,3} + 10H_{-1,0}\zeta_2 + 5H_{-1,-2,0} - 2H_{-1,2,0} +\frac{11}{6}H_0\zeta_2\Big]+(1-x)\Big[\frac{41699}{2592}-3H_{-2;-1,0}-\frac{3}{2}H_{-2}\zeta_2-\frac{128}{9}\zeta_2-4H_{3,0}+\frac{26}{3}\zeta_3-\frac{5}{2}H_{-2,0,0}$ $-7H_{1}\zeta_{2} + \frac{97}{12}H_{1,0,0} + \frac{10}{3}H_{-1,0,0} + \frac{245}{12}H_{3} - 8H_{0,0,0,0} + (1+x) \Big[4H_{3,1} - H_{2,1,1} + \frac{29}{6}H_{-1,2} + \frac{10}{2}H_{-1,0,0} + \frac{245}{12}H_{-1,0,0} + \frac{245}{12}H_{-1,0,0$ $+\frac{17}{6}H_{-2,0}-12H_{2,0}-\frac{31}{12}H_{2,1}+\frac{1}{2}H_{2,0,0}-H_3\zeta_2+\frac{61}{36}H_{1,0}-4H_0\zeta_3-\frac{13}{3}H_{-1}\zeta_2-\frac{46}{3}H_{-1,-1,0}$ $+\frac{\frac{25}{4}H_4+\frac{93}{4}H_9\zeta_2-\frac{55}{9}H_{1,1}-\frac{71}{18}H_2+\frac{49}{18}H_{0,0}-\frac{13}{2}H_{0,0}\zeta_2-\frac{47}{40}\zeta_2^2\Big]+\frac{6131}{2592}-\frac{31}{2}H_{-2}\zeta_2$ $-15H_{-2,-1,0} + \frac{9}{2}H_{-1,0,0} - 3H_{2,1,1} - \frac{9}{4}H_{2,1} + \frac{53}{3}H_{-2,0} - \frac{1}{2}H_{-2,0,0} - 5H_{2,0} - \frac{7}{6}H_{1,1,1} - 8H_0\zeta_3 - 2H_{2,0,0} -$ $-\frac{67}{40}\zeta_2^2 + \frac{29}{6}H_{-1,2} - H_{-1,0} + 8H_{-2,2} + 25H_0\zeta_2 + \frac{412}{9}H_1 + \frac{928}{9}H_0 + \frac{1}{4}H_4 - 65H_3 - 38H_{0,0}$ $-9H_{-3,0} - \frac{17}{3}H_{0,0,0} + \pi \Big[\frac{27}{2}H_{-1,0} - \frac{1}{2}H_{0,0,0,0} + \frac{3}{4}H_{0,0}\zeta_2 + \frac{1}{2}H_{-3,0} - 14H_{0,0,0} + \frac{1}{12}H_{1,1,1} + \frac{1}{12}H_{1,1,$ $-\frac{43}{36}\zeta_2-\frac{1}{2}H_2\zeta_2+\frac{7}{72}H_0+\frac{749}{54}H_1+\frac{135}{4}\zeta_3+\frac{97}{24}H_1\rho+\frac{43}{12}H_1\zeta_2-\frac{85}{12}H_{-1}\zeta_2-\frac{13}{3}H_{1,0,0}$ $+\frac{53}{12}H_{2}+\frac{39}{4}H_{1,1}-2H_{3,1}+\frac{13}{6}H_{-1,-1,0}+\frac{7}{4}H_{2,0,0}-4H_{1,1,0}-4H_{1,2}\Big]\Big)+16C_{p}n_{p}^{-2}\Big(\frac{1}{6}-\frac{1}{9}\frac{1}{\nu}+\frac{1}{9}\frac{1}{2}\frac{1$ $+\frac{2}{9}x - \frac{1}{6}xH_1 + \frac{1}{6}p_{BI}(x)\left[H_{1,1} - \frac{5}{3}H_1\right] + 16C_p^{-2}n_f\left(\frac{4}{9}x^2\left[H_{0,0} - \frac{11}{6}H_0 - \frac{7}{2} + H_{-1,0}\right]\right]$

 $\begin{array}{l} -6H_{1,3}+\frac{49}{4}\zeta_{2}\Big]+\rho_{99}(-x)\Big[\frac{17}{2}H_{-1}\zeta_{3}-\frac{5}{2}H_{-1,-1,0}-\frac{5}{2}H_{-1,2}-\frac{9}{2}H_{-1,0}+\frac{5}{2}H_{-2,0}+\frac{3}{2}H_{-1,0,0}\\ -2H_{3,1}-2H_{4}-6H_{-2,2}+6H_{-2,-1,0}-6H_{-2,0,0}+2H_{0,0}\zeta_{2}+9H_{-3}\zeta_{2}+3H_{-1,-2,0}-2H_{-1,2,1}\\ \end{array}$ $-6H_{-1,-1,-1,0} + 6H_{-1,-1,0,0} + 6H_{-1,-1,2} + 9H_{-1,0}\zeta_2 - 9H_{-1,-1}\zeta_2 - 2H_{-1,2,0} - \frac{11}{2}H_{-1,0,0,0}$ $-6H_{-1,3}$ + $(\frac{1}{v} - x^2)$ $[\frac{55}{12} - 4\zeta_3 + \frac{23}{9}H_{1,0} - \frac{4}{3}H_{1,1,0}]$ + $(\frac{1}{v} + x^2)$ $[\frac{2}{3}H_{1,0,0} - \frac{371}{108}H_1 + \frac{23}{9}H_{1,1,0}]$ $-\frac{2}{2}H_{1,1,1} + (1-x) \left[6H_{2,1,0} + 3H_{2,1,1} - \frac{5}{6}H_{1,1,1} - 7H_{2,0,0} - 2H_{1,2} + 39H_0\zeta_3 - 4H_2\zeta_2 - \frac{16}{2}\zeta_3 + \frac{16}{2}G_{1,2} + \frac{1$ $+H_{1,1,0}+\frac{154}{3}H_0\zeta_2+\frac{899}{24}H_{0,0}+\frac{121}{10}\zeta_2^{-2}+\frac{607}{36}H_2-\frac{5}{2}H_1\zeta_2+\frac{65}{6}H_{1,0,0}-\frac{29}{12}H_{1,0}-\frac{13}{18}H_{1,1}$ $-\frac{1189}{108}H_1 - \frac{67}{3}H_{2,1} - 29H_{2,0} - \frac{949}{36}f_{2,2} - \frac{67}{2}H_{0,0,0} - \frac{142}{3}H_3 + \frac{215}{32} - \frac{3989}{48}H_0 + 2H_{-3,0}$ $+(1+x)\left[H_{-1,0,0}-10H_{-2}\zeta_{2}+6H_{-2,0,0}+2H_{0,0}\zeta_{2}-9H_{-1,-1,0}-7H_{-1,2}-9H_{-2,0}-2H_{3,0}$ $-4H_{-2,-1,0} - 4H_4 - 4H_{3,0} - 4H_{0,0,0,0} + \frac{37}{2}H_{-1,0} + \frac{5}{2}(1+x)H_{-1}\zeta_2 - 4H_{-2,0,0} + 2H_{0,0}\zeta_2$ $+H_2\xi_2 - 3H_{1,1,0} + 2H_{0,0,0,0} + H_{-3,0} - 9H_{2,1,0} - \frac{9}{2}H_{2,1,1} + \frac{11}{2}H_{1,1,1} + \frac{19}{2}H_{2,0,0} + \frac{9}{2}H_{1,2}$ $\begin{array}{c} \frac{2}{-2}H_0\zeta_5 + 8H_{-2}\zeta_2 + \frac{5}{2}H_{-1,-1} + \frac{5}{2}H_{-1,2} + \frac{9}{2}H_{-1,6} + \frac{3}{2}H_{-2,6} + \frac{3}{2}H_{-2,6} + \frac{3}{2}H_{-2,6} + \frac{3}{2}H_{-2,6} + \frac{3}{2}H_{-2,6} + \frac{3}{12}H_{0,5} - \frac{1833}{44}H_{0,6} \\ -\frac{217}{12}\zeta_5 - \frac{9}{2}\zeta_5^2 - \frac{169}{18}H_3 - \frac{13}{14}H_1\zeta_5 - \frac{3}{2}H_{1,6} + \frac{167}{12}H_{1,6} + \frac{109}{18}H_{1,1} + \frac{1283}{12}H_{1,6} + \frac{1283}{12}H_{1,6} + \frac{112}{12}H_{2,6} \\ -\frac{169}{12}H_{-2} + \frac{169}{18}H_{-2} + \frac{13}{12}H_{2,5} - \frac{3}{2}H_{1,6} + \frac{167}{12}H_{1,6} + \frac{109}{18}H_{1,1} + \frac{1283}{12}H_{1,6} + \frac{112}{12}H_{2,6} \\ -\frac{169}{12}H_{-2} + \frac{169}{18}H_{-2} + \frac{16}{12}H_{2,6} + \frac$ $+\frac{75}{+4}H_{2,0}+\frac{170}{9}\zeta_2+\frac{45}{4}H_{0,0,0}+\frac{425}{12}H_3+\frac{7693}{192}+\frac{3659}{48}H_0-2x\left[xH_{2,2}+4H_{3,0}-4H_{-2,3}\right]\right)$ +16 $C_A m_f^2 \left(\frac{1}{6} p_{qg}(x) \left[H_{1,2} - H_1 \zeta_2 - H_{1,0,0} - H_{1,1,0} - H_{1,1,1} - \frac{229}{19} H_0 + \frac{4}{2} H_{0,0} + \frac{11}{2}\right] + x \left[\frac{1}{6} H_2 \right]$ $-\frac{53}{18}H_0 + \frac{17}{6}H_{0,0} - \zeta_3 + \frac{11}{18}\zeta_2 - \frac{139}{108}\Big] + \frac{1}{3}\rho_{08}(-x)H_{-1,0,0} - \frac{53}{162}(\frac{1}{x} - x^2) - \frac{2}{9}(1-x)\Big[6H_{0,0,0}(-x)H_{-1,0,0} - \frac{53}{162}(\frac{1}{x} - x^2) - \frac{2}{9}(1-x)\Big]$ $\frac{7}{6}xH_1 - H_{0,0} + \frac{7}{2}xH_{1,1}\Big] + \frac{7}{9}x(1+x)H_{-1,0} + \frac{7}{4}H_0 - \frac{19}{54}H_1 + H_{0,0,0} + \frac{5}{9}H_{1,1} + \frac{5}{9}H_{-1,0}$ $-\frac{85}{216} + 16C_A^{-2}n_f \left(p_{38}(x) \left[3H_{1,2} + \frac{31}{6}H_{1,0,0} - \frac{17}{2}H_{2,1} + \frac{7}{5}\zeta_2^{-2} - \frac{55}{12}H_{1,1,0} + \frac{31}{12}H_3 - \frac{31}{2}H_1\zeta_3 \right] \right)$ $\frac{216}{12}H_{2,0} - \frac{63}{8}H_{1,0} - \frac{23}{12}H_{1,2} - \frac{155}{6}J_{21} + \frac{25}{24}H_2 - \frac{2537}{27}H_6 + \frac{87}{87} - \frac{23}{2}H_{-1,0,0} + 3H_4 - H_{1,1,1,1} - \frac{155}{6}J_{21} + \frac{25}{24}H_2 - \frac{2537}{27}H_6 + \frac{87}{87} - \frac{23}{2}H_{-1,0,0} + 3H_4 - H_{1,1,1,1} - \frac{155}{6}J_{21} + \frac{25}{2}H_{22} - \frac{155}{2}H_{22} - \frac{155}{2}H_{22}$ $\frac{383}{72}H_{1,1} - \frac{25}{2}H_{-2,0} - \frac{3}{8}\zeta_2 - \frac{7}{4}H_1\zeta_2 - 3H_{0,0}\zeta_2 - \frac{31}{12}H_0\zeta_2 + \frac{103}{216}H_1 + \frac{5}{2}H_{1,0,0,0} + \frac{2561}{72}H_{0,0}$ $+\frac{727}{36}H_{-1,0}-H_{-1}\zeta_2-2H_{-2,2}-\frac{5}{2}H_{-1}\zeta_3-H_{-1,-2,0}+2H_{-1,-1,0,0}+2H_{-1,-1,2}-\frac{3}{2}H_{-1,0,0,0}$ $+6H_{-1,-1,-1,0}-2H_{-1,3}+2H_{-1,2,1}\Big]+(\frac{1}{x}-x^2)\Big[\frac{2}{3}H_{2,1}+\frac{32}{9}\zeta_2-2H_{1,0,0}+\frac{4}{3}H_{1,1,0}-\frac{10}{9}H_{1,1,0}\Big]$ $-\frac{8}{3}H_{-1,0,0}+\frac{3}{2}H_{1,0}+6\zeta_3+\frac{161}{36}H_1-\frac{2351}{108}\Big]+\frac{2}{3}(\frac{1}{x}+x^2)\Big[\frac{26}{3}H_{-1,0}-\frac{28}{9}H_0-2H_{-1,-1,0}$

 $+\frac{1}{3}p_{00}(x)\left[H_{1,2}-H_{1,0}-H_{1}\zeta_{2}+9\zeta_{3}+\frac{83}{12}H_{1,1}+2H_{-2,0}-\frac{7}{36}H_{1}+2H_{0}\zeta_{2}-\frac{1625}{48}+\frac{3}{2}H_{1,0,0}-\frac{3}{2}H_{1,0}+\frac{3}{2}H_{1,0}+\frac{3}{2}H_{1,0}+\frac{3}{2}H_{1,0}+\frac{3}{2}H_{0,0}+\frac{3}$ $+2H_{1,1,0} - \frac{5}{2}H_{1,1,1} + \frac{31}{18}\rho_{09}(-x) \left[\frac{95}{93}H_0 - \zeta_2 - H_{-1,0}\right] + \frac{1}{3}(2-x) \left[6H_{0,0,0,0} - H_3 - \frac{13051}{288}\right]$ $-\frac{13}{2}\xi_{3}-4H_{-2,0}-H_{2,0}-\frac{1}{2}H_{1,0}-\frac{1}{2}H_{2,1}+2H_{0,0,0}-\frac{653}{24}H_{0,0}]+(1+x)\left[H_{0}\xi_{2}-\frac{1187}{216}H_{0}-\frac{1187}$ $+\frac{8}{9}H_2 - \frac{85}{18}H_{-1,0} - \frac{101}{18}\zeta_2 - \frac{80}{27}H_0 + \frac{23}{18}\zeta_2 - \frac{1}{3}H_{1,1} + \frac{5}{4}xH_{1,1} - \frac{1}{9}H_1 - \frac{37}{12}xH_1 + \frac{23}{18}H_{-1,0}$ $+\frac{1501}{54}+H_0\zeta_1-H_{0,0,0}+\frac{101}{3}H_{0,0}-\frac{1}{3}H_{1,0}\Big)+16C_F{}^3\Big(p_{[0]}(x)\Big[3H_{1,1}\zeta_2+3H_1\zeta_2+\frac{7}{2}\zeta_2+2H_1\zeta_3+3H_1\zeta_3+3H_1\zeta_3+\frac{7}{2}G_2+\frac{1}{2}G_2$ $-\frac{23}{8}H_{1,1} - 8H_1\zeta_2 - 6H_{1,-2,0} - 2H_{1,0}\zeta_2 + 3H_{1,1,0} - 3H_{1,1,0,0} - H_{1,1,1,0} + 2H_{1,1,1,1} - 3H_{1,1,2}$ $\frac{6}{-2H_{1,2,0}-2H_{1,2,1}-\frac{9}{2}H_{1,1,1}-\frac{3}{2}H_{1,0,0}-\frac{47}{16}-\frac{47}{16}H_1-\frac{15}{2}\zeta_3 \right]+p_{0,0}(-x)\left[2H_{-1,-2,0}-\frac{15}{2}H_{-1,-2,$ $+6H_{-1,-1,0}+3H_{-1}\zeta_{2}+\frac{7}{4}H_{1,0}-\frac{16}{5}\zeta_{2}^{2}-6H_{-1,0,0}-\frac{7}{2}H_{-1,0}+4H_{-1,-1,0,0}-2H_{-1,0}\zeta_{2}$ $-H_{-1,0,0,0}\Big] + (1-x) \Big[99 H_{1,0,0} + H_{1,1,1} - 10 H_1 \zeta_2 + 3 H_0 \zeta_3 + H_{2,2} - H_2 \zeta_2 + H_{0,0,0} + 5 H_{2,0,0} + 5 H_{2,0,0} + 2 H_{2,0,0} + 2$ $-4H_3+H_{2,1,1}+3H_{0,0}\zeta_2+3H_{3,1}-3H_4+\frac{211}{16}H_1+\frac{49}{20}\zeta_2{}^2\Big]+(1+x)\Big[11\zeta_3+\frac{1}{4}H_{1,1}+\frac{1}{4}H_{1,0}+\frac{1}{4}H_{1,1}+\frac{1}{4}H_{1,0}+\frac{1}{4}H_{1,1}+\frac{1}{4$ $+\frac{91}{16}H_0+36H_{-1,0}+8H_{-1,0,0}-14H_{-1,-1,0}-7H_{-1}\zeta_2+2H_{1,2}+4H_0\zeta_2-H_{2,1}+2H_{-2,0,0}$ $+5H_{-2,0} + \frac{11}{2}H_2 - 2H_{0,0,0,0} - 2H_{-1,-1,0} - H_{-1}\zeta_2 - \frac{13}{4}\zeta_2 + \frac{9}{4}H_{1,0} + \frac{9}{20}\zeta_2^2 + \frac{287}{32} + \frac{11}{16}H_1$ $+4H_{-1,0,0}+16H_{-3,0}-4H_{-2}\zeta_2-8H_{-2,-1,0}-5H_2\zeta_2+\frac{19}{4}H_2+H_{2,2}-\frac{35}{8}H_{0,0}+9H_0\zeta_3$ $+25H_{-2,0}+6H_{-2,0,0}+\frac{3}{2}x\Big[\frac{58}{3}\zeta_{2}-\frac{7}{3}H_{1}\zeta_{2}+4H_{1,1}-\frac{3}{2}H_{1,1,1}+\frac{5}{2}H_{1,0,0}-\frac{175}{06}+H_{3,1}+\frac{19}{3}\zeta_{3}$ $+2H_{2,0}-14H_0+H_{0,0}\zeta_2-H_{-1,0}-H_4-\frac{3}{2}H_{2,1}+\frac{1}{2}H_{2,1,1}+3H_{2,0,0}-\frac{5}{6}H_3-H_{1,2}-\frac{7}{6}H_0\zeta_2$ $+\frac{2}{2}H_{1,1,0}-\frac{29}{6}H_{0,0,0}-\frac{185}{8}H_{0,0}]$

$$\begin{split} & f_{10}^{(0)}(x) = 16C_{x}C_{y}\phi_{y}(x^{2}|\frac{2}{2}R_{1} + 3R_{1}\rho - \frac{27}{12}R_{1} + \frac{3}{8}(L_{-2}\rho - \frac{2}{3}R_{2}L_{1} + \frac{33}{2}L_{-2}\rho - \frac{3}{2}R_{2}L_{1} + \frac{33}{2}L_{-2}\rho - \frac{3}{4}R_{1}L_{1} + \frac{36}{2}L_{2} + 2R_{1}L_{1} + \frac{34}{2}L_{-2}\rho - \frac{3}{4}R_{1}L_{1} + \frac{34}{2}L_{-2}\rho - \frac{3}{4}R_{1}L_{1} + \frac{34}{2}L_{1} + \frac{34}{2}$$

 $-2H_{-1,2} + H_1\zeta_2 + H_{-1}\zeta_2 + \frac{10}{3}H_2 + H_{1,1,1} + (1-x) \left[15H_{0,0,0,0} - 5H_2\zeta_2 - \frac{65}{6}\zeta_3 + \frac{11}{6}H_{1,1,1}\right]$ $-\frac{3}{2}H_4 + \frac{5}{2}H_{0,0}\zeta_2 + H_{1,1,0} - \frac{31}{6}H_{2,0} + \frac{17}{12}H_{1,0} - \frac{551}{20}\zeta_2^{-2} - \frac{29}{4}H_{1,0,0} - \frac{113}{4}H_2 + \frac{18691}{72}H_2$ $\frac{2243}{108} + \frac{265}{6} H_{-1,0,0} + \frac{33}{2} H_{2,0,0} + 19 H_{2,1} + \frac{31}{12} H_{1,1} + \frac{23}{2} H_{-2,0} - \frac{497}{36} I_2' + \frac{29}{6} H_1 \zeta_2 - \frac{143}{12} H_3$ $\frac{11}{-6}H_{1,1,1} - \frac{19}{12}H_0\zeta_2 + \frac{1223}{72}H_1 - \frac{43}{6}H_{0,0,0} - \frac{3011}{36}H_{0,0} \right] + (1+x) \left[8H_{2,1,0} - 4H_{-1,2}\right]$ $+7H_{-1,-1,0} - \frac{35}{6}H_{1,1,1} - 5H_{-2}\zeta_2 - 11H_{-2,0,0} + \frac{1}{3}H_{-1,0} + \frac{15}{2}H_{-1}\zeta_2 + 8H_{3,1} - 10H_{-2,-1,0}$ $+5H_3\zeta_2 + 4H_{2,1,1} - H_{-3,0} + 36H_0\zeta_3 - 5H_2\zeta_2 + 2H_{-1,2} + 6H_{-1,-1,0} - 6H_{2,1,0} - 3H_{2,1,1}$ $-11H_{0,0,0,0} - 5H_{3,1} + \frac{25}{4}H_{1,1,1} + \frac{13}{2}H_{-2}\zeta_2 + \frac{27}{2}H_{-2,0,0} + \frac{11}{2}H_{-3,0} + \frac{13}{2}H_2\zeta_2 - \frac{17}{4}H_{1,0,0}$ $+13H_{-2,-1,0}-\frac{17}{12}H_{1,1,1}-\frac{3}{4}H_4-\frac{1}{4}H_{0,0}\zeta_2+H_{1,2}+\frac{11}{2}H_{1,1,0}+\frac{79}{12}H_{2,0}+\frac{67}{8}H_{1,0}+\frac{263}{8}\zeta_2s^2$ $\begin{array}{c} +\frac{119}{2} \frac{97}{2} +\frac{97}{2} \frac{12}{12} -\frac{355}{12} H_{-1,0} -24H_{0,0} + H_{-} \frac{1}{\sqrt{2}} -\frac{1377}{12} \frac{1}{100} -\frac{1100}{100} -3H_{-1,0,0} -\frac{21}{2} H_{1,0} \\ -\frac{2}{2} H_{2,0,0} -\frac{21}{2} H_{2,0,0} -\frac{21}{2} H_{2,0,0} +\frac{1}{2} H_{1,0,0} +\frac{1}{2} H_{1,0,0} +\frac{1}{12} H_{1,0,0} +$ $-\frac{5}{9}H_{1,1}-\frac{5}{9}H_2-\frac{5}{18}H_{1,0}+\frac{5}{9}J_2'+\frac{1}{6}\rho_{qg}(x)\Big[H_{2,1}+\frac{91}{2}-\frac{35}{3}H_0-\frac{22}{3}H_{0,0}+H_{1,1,1}+6H_{0,0,0}-\frac{1}{3}H_0+\frac{$ $-\zeta_{3}-2\mathfrak{H}_{1,0,0}+\frac{7}{9}H_{1}\Big]+\frac{77}{81}(\frac{1}{x}-x^{2})+(1-x)\Big[\frac{1}{12}H_{1}-\frac{6463}{432}-4H_{0,0,0,0}-\frac{16}{3}H_{0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{2}H_{1,0,0,0,0}+\frac{7}{9}x\mathfrak{H}_{1,1}-\frac{1}{9}x\mathfrak{H}_{1,1$ $+\frac{7}{9}xH_2 + \frac{8}{9}xH_{1,0} - \frac{7}{9}xL_2 - (1+x)\left[\frac{3475}{216}H_0 + \frac{103}{12}H_{0,0}\right] + 16C_F^{2}n_f \left(p_{\rm eg}(x)\left[7H_{1,3} + 7H_4\right] + \frac{103}{12}H_{0,0}\right] + \frac{103}{12}H_{0,0} \left(p_{\rm eg}(x)\left[7H_{1,3} + 7H_4\right] + \frac{103}{12}H_{0,0}\right] + \frac{10}{12}H_{0,0} \left(p_{\rm eg}(x)\left[7H_{1,3} + 7H_4\right] + \frac{10}{12}H_{0,0} \left(p_{\rm eg}(x)\left[7H_{1,3} + 7H_4\right] + \frac{10}{12}H_{0,0}\right] + \frac{10}{12}H_{0,0} \left(p_{\rm eg}(x)\left[7H_{1,3} + 7H_4\right] + \frac{10}{12}H_{0,0}\right] + \frac{10}{12}H_{0,0} \left(p_{\rm eg}(x)\left[7H_{1,3} + 7H_4\right] + \frac{10}{12}H_{0,0} \left(p_{\rm eg}(x)\left[7H_{1,3$ $-2H_{-3,0}-7H_1\zeta_3+5H_{2,2}+6H_{3,0}+6H_{3,1}+H_{2,1,0}+4H_{3,0,0}+3H_{2,1}+2H_{3,1,1}+\frac{5}{2}H_{2,0}$ $+ \frac{61}{8}H_2 - \frac{61}{8}\frac{r_{23}}{r_{3}} + \frac{87}{8}H_1 + \frac{11}{2}H_{1,2} + \frac{61}{8}H_{1,1} + \frac{17}{2}H_{1,0} - 7H_{6,0}r_{2} + \frac{5}{2}H_{1,0,0} + \frac{5}{2}H_{1,1,0} - \frac{19}{2}r_{23}^{\prime} \\ + \frac{81}{32} + \frac{12}{2}H_3 - \frac{11}{2}H_0r_{23}^{\prime} - \frac{7}{2}H_1r_{22} + \frac{15}{2}H_{0,0,0} + \frac{8}{8}H_0 + \frac{11}{3}r_{23}^{\prime} + 3H_{1,1,1} - 5H_2r_{22} - 7H_0r_{23}^{\prime} \\ + \frac{81}{32} + \frac{12}{3}H_{1,1,1} - 5H_2r_{22} - 7H_0r_{23}^{\prime} + \frac{15}{3}H_{0,0,0} + \frac{8}{3}H_0 + \frac{11}{3}r_{23}^{\prime} + 3H_{1,1,1} - 5H_2r_{22} - 7H_0r_{23}^{\prime} \\ + \frac{81}{32} + \frac{12}{3}H_{0,0} + \frac{$ $+11H_{0,0}-2H_{1,-2,0}-7H_{1,0}\zeta_{2}+3H_{1,0,0,0}-5H_{1,1}\zeta_{2}+4H_{1,1,0,0}+H_{1,1,1,0}+2H_{1,1,1,1}+5H_{1,1,1,0}+2H_{1,1,1,1}+5H_{1,1,1,1}+2H_{1,1,1,1}+5H_{1,1,1,1}+2H_{1$ $+6H_{1,2,0} + 6H_{1,2,1} + 4p_{4g}(-x) \left[H_{0,0,0} - H_{-2,0} + H_{-1,-1,0} - H_{-2,0,0} + \frac{1}{2}H_{-1,-2,0} - \frac{5}{8}H_{-1,0}\right]$ $-\frac{5}{4}H_{-1,0,0} - \frac{1}{2}H_{-3,0} + \frac{1}{2}H_{-1}\zeta_2 + H_{-1,-1,0,0} - \frac{1}{4}H_{-1,0,0,0}\Big] + 2(1-x)\Big[H_{2,1,0} - H_{2,0,0} - H_{2,2} - H_{2,0,0} - H_{2,0$ $-H_{3,1}-2H_{3,0}-2H_{-1}\zeta_2+H_{1,2}-H_{1,0,0}-H_{1,1,0}+H_2\zeta_2-\zeta_2^{-2}+\frac{43}{8}H_2+\frac{49}{8}\zeta_2+\frac{13}{8}H_{1,1}$ $-\frac{33}{16}H_1+\frac{5}{2}H_{1,0}+\frac{7}{2}H_{0,0}\zeta_2+\frac{21}{4}\zeta_3+\frac{479}{64}-\frac{1}{2}H_{1,1,1}-\frac{1}{2}H_3+\frac{1}{4}H_{2,1}+\frac{1}{2}H_{2,1,1}+\frac{3}{2}H_0\zeta_2$ $+\frac{1}{2}H_0\zeta_1 - \frac{7}{2}H_4 + H_1\zeta_2 - \frac{19}{2}H_{0,0,0} - \frac{239}{16}H_{0,0} - \frac{405}{32}H_0 + 8(1+x)[H_{-1,-1,0} - H_{-1,0,0}]$

 $+12H_{0,0,0,0} - \frac{293}{108} + \frac{61}{6}H_0\zeta_2 - \frac{7}{3}H_{1,0} - \frac{857}{36}H_1 - 9H_0\zeta_3 + 16H_{-2,-1,0} - 4H_{-2,0,0} + 8H_{-2}\zeta_2$ $-\frac{13}{2}H_{1,0,0} + \frac{3}{4}H_{1,1} - H_{1,1,0} - H_{1,1,1} + (1+x) \left[\frac{1}{6}H_{2,0} - \frac{95}{3}H_{-1,0} - \frac{149}{36}H_2 + \frac{3451}{100}H_0 + \frac{149}{100}H_2 +$ $-7H_{-2,0} + \frac{302}{9}H_{0,0} + \frac{19}{6}H_3 - \frac{991}{36}\zeta_3 - \frac{163}{6}\zeta_3 - \frac{35}{3}H_{0,0,0} + \frac{17}{6}H_{2,1} - \frac{43}{10}\zeta_3^2 + 13H_{-1}\zeta_2$ $+18H_{-1,-1,0}-H_{3,1}-6H_4-4H_{-1,2}+6H_{0,0}\zeta_2+8H_2\zeta_2-7H_{2,0,0}-2H_{2,1,0}-2H_{2,1,1}-4H_{3,0}\zeta_2+8H_2\zeta_2-7H_{2,0,0}-2H_{2,1,0}-2H_{2,1,1}-4H_{3,0}\zeta_2+8H_2\zeta_2-7H_{2,0,0}-2H_{2,1,0}$ $-9\mathbf{H}_{-1,0,0}\Big]-\frac{241}{248}\delta(1-x)\Big)+16C_{A}n_{f}^{2}\Big(\frac{19}{54}\mathbf{H}_{0}-\frac{1}{24}\mathbf{x}\mathbf{H}_{0}-\frac{1}{27}p_{\mathbf{0}\mathbf{0}}(x)+\frac{13}{54}[\frac{1}{x}-x^{2})\Big[\frac{5}{3}-\mathbf{H}_{1}\Big]$ $+(1-x)\left[\frac{11}{72}H_1-\frac{71}{216}\right]+\frac{2}{9}(1+x)\left[\zeta_2+\frac{13}{12}xH_0-\frac{1}{2}H_{0,0}-H_2\right]+\frac{29}{288}\delta(1-x)\right)$ $+16 C_{A}^{2} n_{f} \left(x^{2} \left[\zeta_{3}^{'}+\frac{11}{9} \zeta_{2}^{'}+\frac{11}{9} H_{0,0}-\frac{2}{3} H_{3}+\frac{2}{3} H_{0} \zeta_{2}^{'}+\frac{1639}{108} H_{0}-2 H_{-2,0}\right]+\frac{1}{3} p_{88}(x) \left[\frac{10}{3} \zeta_{2}^{'}+\frac{1}{9} H_{0,0}-\frac{2}{3} H_{0,0}-\frac{2$ $-\frac{209}{36} - 8\zeta_3 - 2H_{-2,0} - \frac{1}{2}H_0 - \frac{10}{3}H_{0,0} - \frac{20}{3}H_{1,0} - H_{1,0,0} - \frac{20}{3}H_2 - H_3 + \frac{10}{9}\rho_{gg}(-x) \left[\zeta_3 - 2H_{-2,0} - \frac{1}{3}H_{0,0} - \frac{10}{3}H_{0,0} - \frac{$ $\begin{array}{c} {}_{50} \\ +2H_{-1,0}+\frac{3}{10}H_0\zeta_2-H_{0,0} \right] +\frac{1}{3}(\frac{1}{x}-x^2) \left[H_3-H_0\zeta_2-\frac{13}{3}H_2+\frac{5443}{108}-3H_1\zeta_2+\frac{25}{36}H_1 \right] \\ -\frac{13}{3}H_1\rho+H_{1,0,0} \right] +(\frac{1}{x}+x^2) \left[\frac{151}{54}H_0-\frac{3}{3}\zeta_2+\frac{1}{3}H_{-1}\zeta_2-\zeta_3+2H_{-1,-1,0}-\frac{2}{3}H_{-1,0,0} \right] \end{array}$ $-\frac{37}{9}H_{-1,0} + \frac{2}{3}H_{-1,2}\Big] + (1-x)\Big[\frac{5}{6}H_{-2,0} + H_{-3,0} + 2H_{0,0,0} - \frac{269}{36}\zeta_2 - \frac{4097}{216} - 3H_{-2}\zeta_2$ $-6H_{-2_{1}-1_{0}}+3H_{-2_{1}0,0}-\frac{7}{2}H_{1}L_{2}^{\prime}+\frac{677}{72}H_{1}+H_{1,0}+\frac{7}{4}H_{1,0,0}\Big]+(1+x)\Big[\frac{193}{36}H_{2}-\frac{11}{2}H_{-1}L_{2}^{\prime}$ $+\frac{39}{20}\zeta_{2}^{2} - \frac{7}{12}H_{3} - \frac{53}{9}H_{0,0} + \frac{7}{12}H_{0}\zeta_{2} - \frac{5}{2}H_{0,0}\zeta_{3} + 5\zeta_{3} - 7H_{-1,-1,0} + \frac{77}{6}H_{-1,0} + \frac{9}{2}H_{-1,0,0}$ $+2H_{-1,2} - 3H_2\zeta_2 - \frac{2}{3}H_{2,0} + \frac{3}{2}H_{2,0,0} + \frac{3}{2}H_4 + \frac{1}{9}\zeta_2 + 7H_{-2,0} + 2H_2 + \frac{458}{27}H_0 + H_{0,0}\zeta_2$ $+\frac{3}{2}\zeta_{2}^{2}^{2}+4H_{-3,0}-x\Big[\frac{131}{12}H_{0,0}-\frac{8}{3}H_{0}\zeta_{2}+\frac{7}{2}H_{3}-H_{0,0,0,0}+\frac{7}{6}H_{0,0,0}+\frac{1943}{216}H_{0}+6H_{0}\zeta_{3}\Big]$ $-\delta(1-x)\left[\frac{233}{288}+\frac{1}{6}\zeta_{2}+\frac{1}{12}\zeta_{2}^{2}+\frac{5}{3}\zeta_{3}\right]+16C_{A}^{-3}\left(x^{2}\left[33H_{-2,0}+33H_{0}\zeta_{2}-\frac{1249}{18}H_{0,0}-\frac{1249}{18}H_$ $-44H_{3,0,0} - \frac{110}{3}H_3 - \frac{44}{3}H_{2,0} + \frac{35}{6}\zeta_3 + \frac{6409}{108}H_0 + p_{88}(x) \left[\frac{245}{24} - \frac{67}{9}\zeta_3 - \frac{3}{10}\zeta_3^2 + \frac{11}{3}\zeta_3 + \frac{11}$ $-4H_{-3,0} + 6H_{-2}\zeta_2 + 4H_{-2,-1,0} + \frac{11}{3}H_{-2,0} - 4H_{-2,0,0} - 4H_{-2,2} + \frac{1}{6}H_0 - 7H_0\zeta_3 + \frac{67}{9}H_{0,0}$ $-8H_{0,0}\zeta_2 + 4H_{0,0,0,0} - 6H_1\zeta_3 - 4H_{1,-2,0} + 10H_{2,0,0} - 6H_{1,0}\zeta_2 + 8H_{1,0,0,0} + 8H_{1,1,0,0} + 8H_{4,1,0,0} + 8H_{4,1,0,0} + 8H_{4,1,0,0,0} + 8H_{4,1,0,0,0}$ $\frac{134}{9}H_{1,0} + \frac{11}{6}H_{1,0,0} + 8H_{1,2,0} + 8H_{1,3} + \frac{134}{9}H_2 - 4H_2\zeta_2 + 8H_{3,1} + 8H_{2,2} + \frac{11}{6}H_3 + 10H_{3,0}$ $+8H_{2,1,0}$] + $p_{gg}(-x)$ [$\frac{11}{2}\zeta_2^2 - \frac{11}{6}H_0\zeta_2 - 4H_{-3,0} + 16H_{-2}\zeta_2 - 12H_{-2,2} - \frac{134}{9}H_{-1,0} + 2H_2\zeta_2$ $\begin{array}{c} +8H_{-2,-1,0}+12H_{-1}\zeta_{3}-18H_{-2,0,0}+8H_{-1,-2,0}-16H_{-1,-1}\zeta_{2}+24H_{-1,-1,0,0}+16H_{-1,-1,2}\\ +18H_{-1,0}\zeta_{3}-16H_{-1,0,0,0}-4H_{-1,2,0}-16H_{-1,3}-5H_{0}\zeta_{3}-8H_{0,0,0}\zeta_{2}+4H_{0,0,0,0}+2H_{3,0}\\ \end{array}$ $-\frac{67}{9}\zeta_2 + \frac{67}{9}H_{0,0} + 8H_4 \Big] + \Big(\frac{1}{x} - x^2\Big) \Big[\frac{16619}{162} + \frac{22}{3}H_{2,0} - \frac{55}{2}\zeta_3 - \frac{11}{2}H_0\zeta_3 - \frac{67}{9}H_2 - \frac{67}{9}H_1 - \frac{67}{9}H_1 - \frac{67}{9}H_2 - \frac{67}{9}H_2$

$$\begin{split} &-H_{0,\beta,\rho,h}+\frac{4}{3}H_{-2,\rho}-\frac{4}{3}H_{-1,\rho}\right)-4H_{-1,-(h,+)}+3H_{0,\beta,\rho,h}+3H_{-1,\beta,\rho}-1H_{-2,\beta}+\frac{1}{3}H_{-1,\rho}\\ &+44H_{-2,\beta,\rho}-\frac{113}{3}H_{2}-\frac{17}{3}\xi_{2,\sigma}-\frac{2}{3}H_{1}-\frac{1}{3}H_{2,\sigma}-\frac{3}{3}H_{1,\sigma}-\frac{7}{3}H_{1,\sigma}-\frac{7}{3}H_{2,\sigma}\xi_{2,\sigma}-\frac{5}{3}H_{1,\sigma}\xi_{2,\sigma}-\frac{5}{3}H_{1,\sigma}-\frac{1}{3}H_{2,\sigma}-\frac{1}{3}H_{2,\sigma}-\frac{1}{3}H_{2,\sigma}-\frac{1}{3}H_{2,\sigma}-\frac{1}{3}H_{2,\sigma}-\frac{1}{3}H_{2,\sigma}-\frac{1}{3}H_{2,\sigma}-\frac{1}{3}H_{2,\sigma}-\frac{1}{3}H_{2,\sigma}-\frac{1}{3}H_{2,\sigma}-\frac{1}{3}H_{2,\sigma}-\frac{1}{3}H_{2,\sigma}-\frac{1}{3}H_{2,\sigma}-\frac{1}{3}H_{2,\sigma}-\frac{1}{3}H_{2,\sigma}-\frac{1}{3}H_{2,\sigma}+\frac{1}{3}H_$$

 $P_{gq}^{(2)}(\mathbf{x}) = 16C_A C_p n_f \left(\frac{2}{9} \kappa^2 \left[\frac{25}{6} H_1 - \frac{131}{4} + 3\zeta_2 - H_{-1,0} - 3H_2 + H_{1,1} + \frac{125}{6} H_0 - H_{0,0}\right]\right)$ $+\frac{5}{6}p_{00}(x)\left[H_{1,2}+H_{2,1}+\frac{967}{120}+\frac{251}{90}H_1-\frac{39}{10}H_{1,1}-3\zeta_3-\frac{2}{5}H_0\zeta_2-\frac{1}{5}H_1\zeta_2-\frac{4}{3}H_{1,0}+H_{1,1,0}\right]$ $-\frac{2}{5}H_{1,0,0}+H_{1,1,1}+\frac{2}{5}H_{2,0}\Big]+\frac{2}{3}p_{gq}(-x)\Big[2H_{-1}\zeta_{2}+\frac{7}{4}\zeta_{2}+\frac{41}{12}H_{-1,0}-\frac{151}{72}H_{0}+\frac{1}{2}H_{-2,0}$ $+\frac{5}{2}H_2 + 2H_{-1,-1,0} - H_{-1,0,0} - H_{-1,2} + \frac{2}{3}(1-x)[H_{-2,0} + 2\zeta_3 - H_3] + (1+x)[\frac{179}{100}H_1]$ $+\frac{5}{9}\zeta_2 + \frac{25}{9}H_{-1,0} - \frac{5}{36}H_{1,1} - \frac{167}{36}H_{0,0} - \frac{1}{3}H_{2,1} - \frac{4}{3}H_0\zeta_2 - \frac{193}{72} + \frac{1}{4}H_1 + \frac{1}{9}H_{-1,0} + 4H_2$ $-\frac{1}{4}H_{1,1}+\frac{227}{18}H_{0}-\frac{35}{12}H_{0,0}-H_{2,1}-\frac{2}{3}H_{0}\zeta_{2}+\frac{10}{3}H_{-2,0}+3\zeta_{3}+2H_{3}+\frac{2}{3}H_{0,0,0}+x\Big[\frac{11}{4}\zeta_{2}$ $-\frac{523}{144} - \frac{19}{36}H_2 + \frac{271}{108}H_0 - \frac{5}{6}H_{1,0} + 16C_4C_F^2 \left(x^2 \left[\frac{7}{2} + \frac{173}{54}H_1 - 2\zeta_3 - \frac{2}{3}H_{1,1,1} - \frac{26}{9}H_{1,1}\right] + \frac{26}{9}H_{1,1,1} - \frac{26}{9}H_{1,1} - \frac{26}{9}H_{1,1,1} - \frac{26}{9}H_{$ $-6H_2 + 2H_{2,1} + 6\zeta_2 + \frac{335}{54}H_0 - \frac{28}{9}H_{0,0} - \frac{8}{3}H_{0,0,0} \Big] + p_{gq}(x) \Big[\frac{3}{2}H_1\zeta_3 + \frac{163}{32} - 5\zeta_2 + \frac{27}{4}\zeta_3 + \frac{163}{32} + \frac{$ $+\frac{6503}{432}H_1+\frac{2}{9}H_{1,1}+\frac{35}{3}H_{1,1,1}+4H_2+\frac{9}{2}H_{2,1}+4H_{1,0,0}+2H_{2,0,0}-H_2\zeta_2+\frac{41}{12}H_{1,2}+H_{2,2}$ $+\frac{191}{24}H_{1,0}+3H_{2,0}-2H_{2,1,1}-\frac{3}{2}H_{-1}\zeta_2-\frac{59}{12}H_1\zeta_2+5H_{1,-2,0}+H_{1,0}\zeta_2+\frac{5}{2}H_{1,0,0,0}-2H_{1,1}\zeta_2$ $+\frac{1}{12}H_{1,1,0} + 5H_{1,1,0,0} - 3H_{1,1,1,0} - 4H_{1,1,1,1} - H_{1,1,2} - 2H_{1,2,1} + H_{2,1,0} + p_{gq}(-x)[H_{-1,0}]$ $+H_{-1,0}\zeta_2 + \frac{3}{2}H_{-1,0,0} + \frac{27}{10}\zeta_2^2 - 3H_{-1,-1,0} - \frac{11}{2}H_{-1}\zeta_3 - 3H_{-1,-2,0} - \frac{3}{2}H_{-1,0,0,0} - 3H_{-1,2}$ $+5H_{-1,-1}\zeta_2 - 4H_{-1,-1,0,0} - 2H_{-1,-1,2} + 6H_{-1,-1,-1,0} + 2H_{-1,2,1}$ + $(1-x)[H_2\zeta_2 - H_{2,2}]$ $+\frac{23}{12}H_{1,0}-\frac{7061}{432}H_{0}-\frac{4631}{144}H_{0,0}-\frac{38}{3}H_{0,0,0}-H_{-3,0}-2H_{3,0}-\frac{4433}{432}H_{1}-2H_{2,0,0}-\frac{21}{2}H_{1,0,0}$ $-\frac{2}{5}\zeta_{2}^{2}-\frac{7}{2}H_{1,2}+\frac{23}{2}H_{1}\zeta_{2}-4H_{0}\zeta_{3}\Big]+(1+x)\Big[\frac{49}{6}H_{3}-H_{-2,0}-\frac{55}{6}H_{0}\zeta_{2}-\frac{1}{2}H_{3,1}-\frac{1159}{36}\zeta_{2}$ $\frac{\frac{655}{1576} - \frac{151}{6}\zeta_3 - \frac{185}{18}H_{1,1} + \frac{1}{6}H_{1,1,1} - \frac{95}{9}H_2 + \frac{29}{6}H_{2,1} - \frac{171}{4}H_{-1,6} - 12H_{-1,6,6} + 7H_{-1}\zeta_2}$ $+16H_{-1,-1,0} + \frac{5}{2}H_{2,0} + \frac{3}{2}H_{2,1,1} + 4H_{0,0,0,0} - 35H_{-2,0} - \frac{179}{22}H_0 + \frac{2041}{144}H_{0,0} - \frac{19}{6}H_{0,0,0}$

 $-\frac{413}{108}H_1 - \frac{11}{2}H_1\zeta_2 + \frac{33}{2}H_{1,0,0} + 11(\frac{1}{x} + x^2)\left[\frac{71}{54}H_0 - \frac{1}{6}H_3 - \frac{389}{198}\zeta_2 - \frac{2}{3}H_{-3,0} - \frac{1}{2}H_{-1}\zeta_2\right]$ $+H_{-1,-1,0} - \frac{523}{198}H_{-1,0} + \frac{8}{3}H_{-1,0,0} + H_{-1,2} + (1-z) \left[\frac{31}{36}H_1 + \frac{27}{2}H_{1,0} - \frac{25}{2}H_{1,0,0} - 4H_{-3,0}\right]$ $-\frac{263}{12}H_{0,0} - \frac{29}{3}H_{0,0,0} - \frac{19}{3}H_{-3,0} - \frac{11317}{108} - 4H_{-2}\zeta_2 - 8H_{-3,-1,0} - 12H_{-2,0,0} - \frac{3}{2}H_1\zeta_2$ $+(1+x)\left[\frac{27}{2}H_{0}\zeta_{2}-\frac{43}{6}H_{3}+\frac{29}{3}H_{2,0}+\frac{4651}{216}H_{0}-\frac{329}{18}\zeta_{2}+\frac{11}{2}(1+x)\zeta_{3}-\frac{43}{5}\zeta_{2}^{2}-\frac{215}{6}H_{-1,0}-\frac{10}{2}H_{$ $-22H_{0,0}\zeta_{3}-8H_{0}\zeta_{3}-3H_{-1,-1,0}+38H_{-1,0,0}+25H_{-1,3}+10H_{2,0,0}-4H_{2}\zeta_{2}+16H_{3,0}+26H_{4}$ $\frac{-158}{9}H_2 - \frac{53}{2}H_{-1}\zeta_2 \Big] - 29H_{0,0} - \frac{40}{3}H_{0,0,0} + 27H_{-1,0} + \frac{41}{3}H_0\zeta_2 - 20H_3 - 24H_{2,0} + \frac{53}{5}\zeta_2$ $+\frac{601}{12}H_0 + 24\zeta_3 + 2\zeta_2^2 + 27H_2 - 4H_0\rho\zeta_3 - 16H_0\zeta_3 - 16H_{-3,0} + 28xH_{0,0,0,0} + 8(1-x)\left[\frac{79}{32}\right]$ $-\zeta_{3}\zeta_{5} + \frac{1}{z}\zeta_{2} + \frac{11}{24}\zeta_{2}^{2} + \frac{67}{6}\zeta_{5} - 5\zeta_{5}\right) + 16C_{F}n_{f}^{-2}\left(\frac{2}{5}x^{2}\left[\frac{11}{6}H_{0} + H_{2} - \zeta_{3} + 2H_{0,0} - 9\right] + \frac{1}{3}H_{2}$ $\begin{array}{c} -\frac{1}{3}\zeta_{2}-\frac{10}{9}H_{0}-\frac{1}{3}H_{0,0}+2+\frac{2}{9}(\frac{1}{x}-x^{2})\left[\frac{8}{3}H_{1}-2H_{1,0}-H_{1,1}-\frac{77}{18}\right]-(1-x)\left[\frac{1}{3}H_{1,0}+\frac{1}{6}H_{1,1}-\frac{1}{9}H_{1,1}-\frac$ $-H_{2,1}-2H_{2,0}\Big]+\frac{11}{144}\delta(1-x)\Big)+16C_{p}^{-2}n_{f}\Big(\frac{4}{3}x^{2}\Big[\frac{163}{16}+\frac{27}{8}H_{0}+\frac{7}{2}H_{0,0}-H_{2,0}-\zeta_{2}+\frac{9}{4}H_{1,0}-H_{2,0}-H_{2,0}-\zeta_{2}+\frac{9}{4}H_{1,0}-H_{2,0}-H_{2,0}-H_{2,0}-\zeta_{2}+\frac{9}{4}H_{1,0}-H_{2,0}-H$ $-H_{2,1} + \frac{1}{2}H_{0,0,0} + \frac{85}{16}H_1 + H_2 - 2H_{-2,0} - \frac{3}{2}\zeta_3 \Big] + \frac{4}{3}(\frac{1}{x} - x^2) \Big[\frac{31}{16}H_1 - \frac{11}{16} - \frac{5}{4}H_{1,0} + \frac{1}{2}H_{1,0,0} - \frac{3}{2}H_{1,0} \Big] + \frac{1}{3}H_{1,0,0} + \frac{1}{3}H_{1,0$ $-H_1\zeta_2 - H_{1,1} + H_{1,1,0} + H_{1,1,1} + \zeta_3 + \frac{4}{3}(\frac{1}{x} + x^2)[H_{-1}\zeta_2 + 2H_{-1,-1,0} - H_{-1,0,0}] + \frac{215}{12}H_{0,0}$ $+\frac{20}{3}H_{0}-\frac{131}{6}+3H_{2,0}+\frac{205}{12}H_{2,2}-3H_{1,0}+H_{2,1}-\frac{85}{12}H_{1}+\frac{11}{4}H_{2}+8H_{-2,0}+2f_{2}^{-2}-H_{0}f_{2}$ $+H_3 + 6H_0\zeta_3 + 8H_{-3,0} - 4\pi H_{0,0,0} + (1-x) \Big[\frac{107}{12} H_1 - \frac{5}{6} H_{1,0} - 4\zeta_2 + H_0\zeta_3 - 8H_{-2,-1,0} \Big]$ $-4H_{-2}\zeta_{2}+4H_{-2,0,0}-4H_{1}\zeta_{2}+\frac{7}{2}H_{1,0,0}-\frac{7}{12}H_{1,1}+H_{1,1,0}+H_{1,1,1}\right]+(1+x)\left[\frac{5}{4}H_{2}+\frac{33}{8}H_{2}$ $-\frac{99}{4}H_{0,0}-8H_{2,0}-\frac{541}{24}H_0-4H_{2,1}-\frac{3}{2}H_{0,0,0}-2x\zeta_3+\frac{9}{2}\zeta_2{}^2+5H_0\zeta_3-5H_3-4H_{-1}\zeta_2$ $-8H_{-1,-1,0} + \frac{67}{2}H_{-1,0} + 4H_{-1,0,0} + 2H_{0,0}\zeta_2 - 2H_{0,0,0,0} - 4H_2\zeta_2 + 3H_{2,0,0} + 2H_{2,1,0}$ $+2H_{2,1,1}+H_{3,1}-2H_4 + \frac{1}{16}\delta(1-n)$.

Moch, Vermaseren, Vogt **2004**

NNLO the new emerging standard in QCD – essential for precision physics

You may be wondering,

"Am I in the right place?"

What does this have to do with NUNCLEAR PHYSICS?

What is a nucleus? depends on the resolution_T scale!

A point particle: $\lambda_t >> 10$ fm

A collection of nucleons: $\lambda_t \sim 1 \text{ fm}$

A system of <u>quarks and gluons</u>: $\lambda_t \ll 1$ fm

we should be able to use pQCD





modification of the nuclear structure functions



 $(\propto 1/c.m.\,energy)$

How about scattering of nuclei? RHIC, LHC

I) modification of initial state: "nuclear shadowing"

II) modification of hard scattering: multiple scattering

III) modification of fragmentation functions

how do partons hadronize?

so far we have considered PQCD in the Bjorken limit $Q^2, S \to \infty x_{Bj} \equiv \frac{Q^2}{S}$ fixed

DGLAP evolution of partons number of partons increases with Q² but parton number density decreases hadron becomes more dilute

Excellent tool for high Q² inclusive observables higher twists become important at low Q²

Not designed to treat collective phenomena: shadowing multiple scattering diffraction impact parameter dependence

Extension beyond leading twist is very difficult many-body dynamics hidden in parameters

