Hadronic physics (interactions) with Lattice QCD

${}^{A}_{Y}Z$

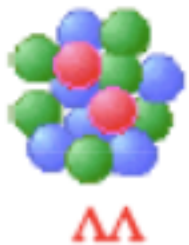
hyperons in nuclei:

- ✓distinguishible from nucleons
- ✓glue-like role
- ✓new spectroscopy
- ✓source of information about the strong $\Lambda N \to \Lambda N$ and weak $\Lambda N \to NN$ interactions

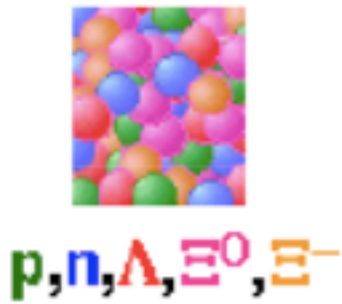
[there are no stable hyperon beams –unstable against the weak interaction-]

S=0

S=-1



S=-2

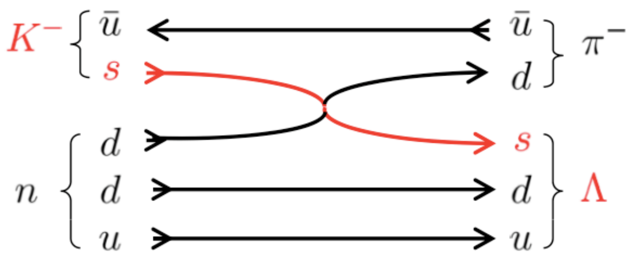


…

PRODUCTION REACTIONS

Strangeness exchange: $n(K^-,\pi^-)\Lambda$
$p(K^-,\pi^\pm)\Sigma^\mp$

CERN, BNL,KEK
FINUDA@DAPHNE

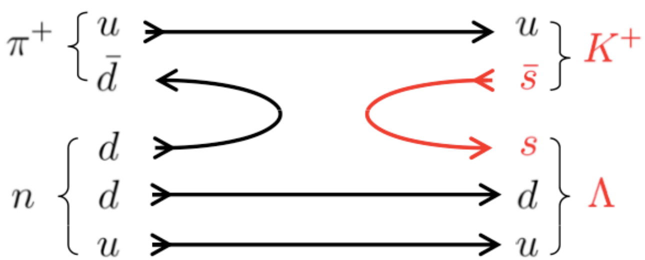


$$K^- + {}^{12}C \to \pi^- + {}^{12}_{\Lambda}C$$

${}^{A}_{\Lambda}Z$

Associated production: $n(\pi^+,K^+)\Lambda$

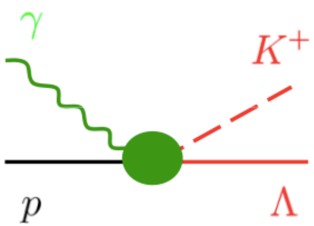
BNL, KEK



$$\pi^+ + {}^{12}C \to K^+ + {}^{12}_{\Lambda}C$$

Electroproduction: $p(\gamma,K^+)\Lambda$
$p(e,e'K^+)\Lambda$

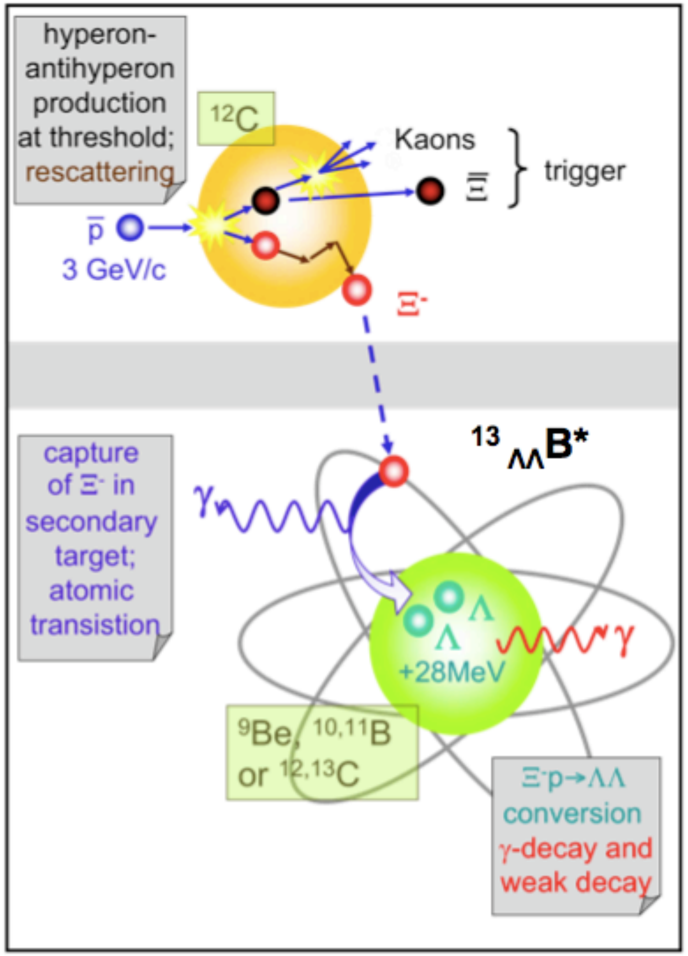
Jlab,MAMI-C



$$e + {}^{12}C \to e' + K^+ + {}^{12}_{\Lambda}B$$

$^{A}_{\Lambda\Lambda}Z$

PANDA detector @ FAIR



- $\bar{p}$ +Nucleus->Ξ^- +Ξ^+ at 3GeV/c
  Other Exp. E906 AGS-BNL, JPARC
  (K^- + p -> K^+ + Ξ^-)

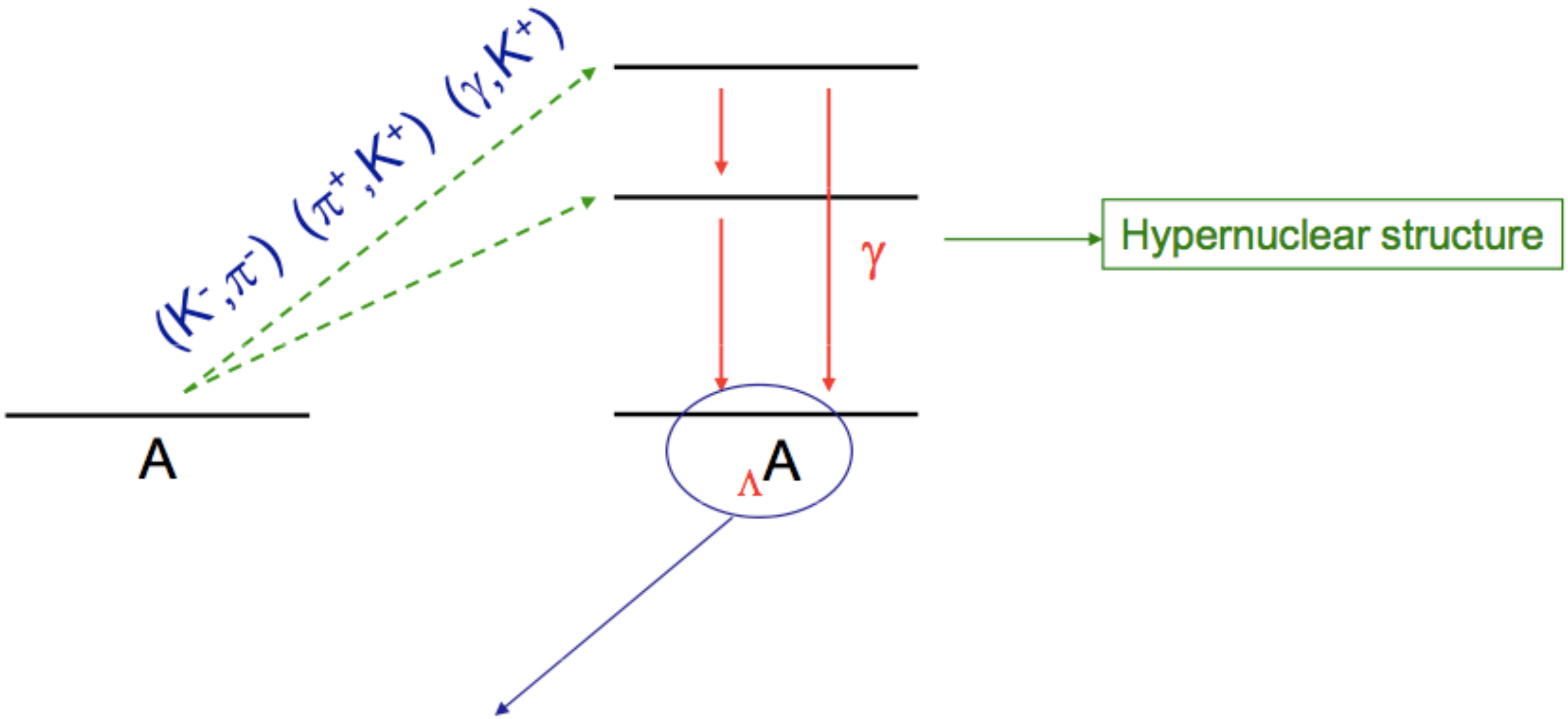
- Cross section 2μb

- Luminosity 10^{32} cm^{-2}/s to
  7.10^5 Ξ^- +Ξ^+ hour

- Ξ^- p -> ΛΛ + 28 MeV
- energy release may give rise to the emission of excited hyperfragments ($^{13}_{\Lambda\Lambda}B^*$)

- Two-step production mechanism requires a
  1. devoted setup
  2. spectroscopy: decay products

WEAK HYPERNUCLEAR DECAY



Weak decay of hypernuclei allows to obtain new and complementary information on the properties of hypernuclei and the weak YN interaction

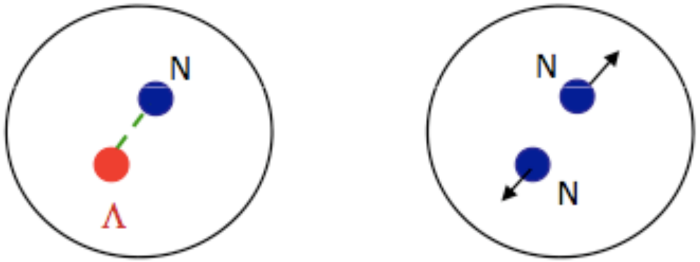
WEAK HYPERNUCLEAR DECAY

MESONIC

Γ_{π^-} : $\Lambda \rightarrow \pi^- p$

$\Gamma_{\pi 0}$: $\Lambda \rightarrow \pi^0 n$

k_N~100 MeV/c < k_F



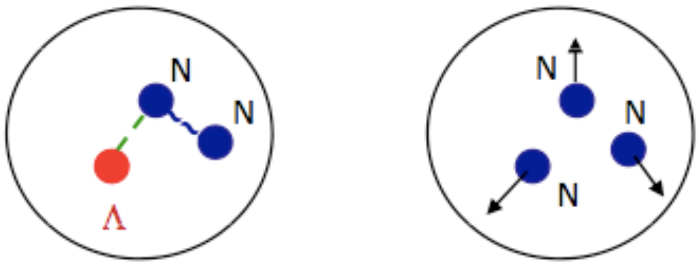
NON-MESONIC

dominant for $A \geq 5$
WHY?

Γ_n: $\Lambda n \rightarrow n n$

Γ_p: $\Lambda p \rightarrow n p$

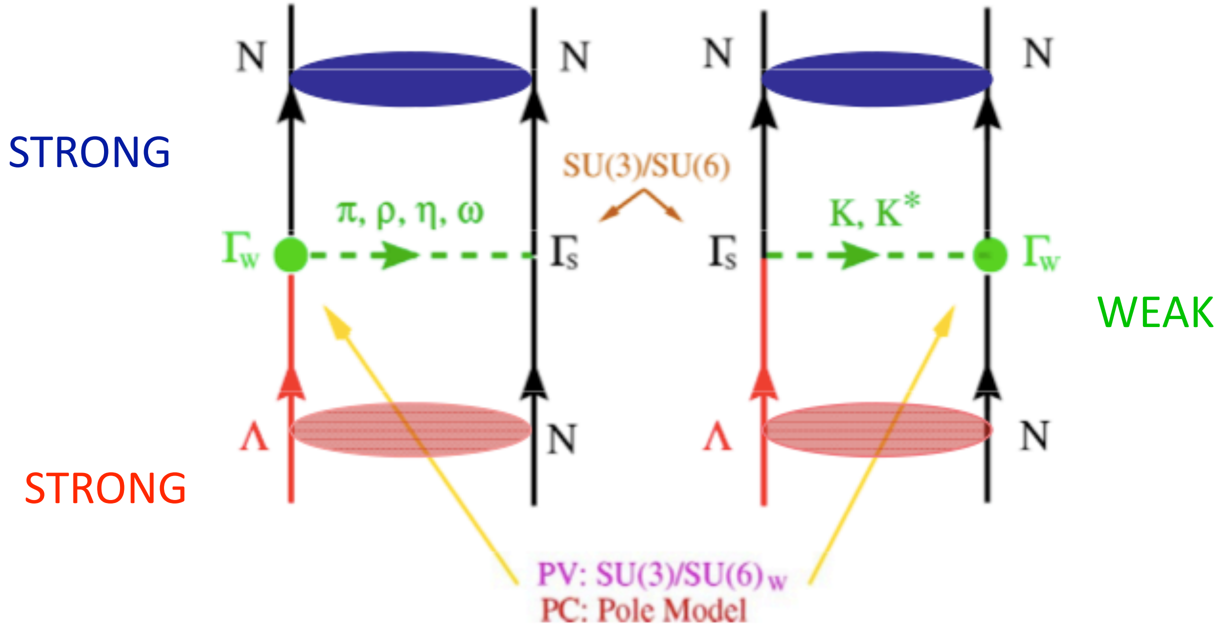
k_N~400 MeV/c



Γ_2: $\Lambda N N \rightarrow n N N$ k_N~340 MeV/c

$$\Gamma_T = \Gamma_M + \Gamma_{NM} = \Gamma_{\pi^-} + \Gamma_{\pi 0} + \Gamma_n + \Gamma_p + \Gamma_2$$

If we want to use extract information about the |ΔS|=1 $\Lambda N \to NN$ from hypernuclear decay, we will need to have some control of the strong interaction among hadrons



Weak interaction: $B_1 \to B_2 M$ ← From the transitions $\Lambda \to \pi N$, $\Sigma \to \pi N$, using SU(3)/SU(6)

Strong interaction: $B_1 \to B_2 M$

Initial state correlations $\boxed{\Lambda N}$

Final state correlations NN

} ← From realistic BB forces: Nijmegen89, Julich89

how well is this known?

more motivation

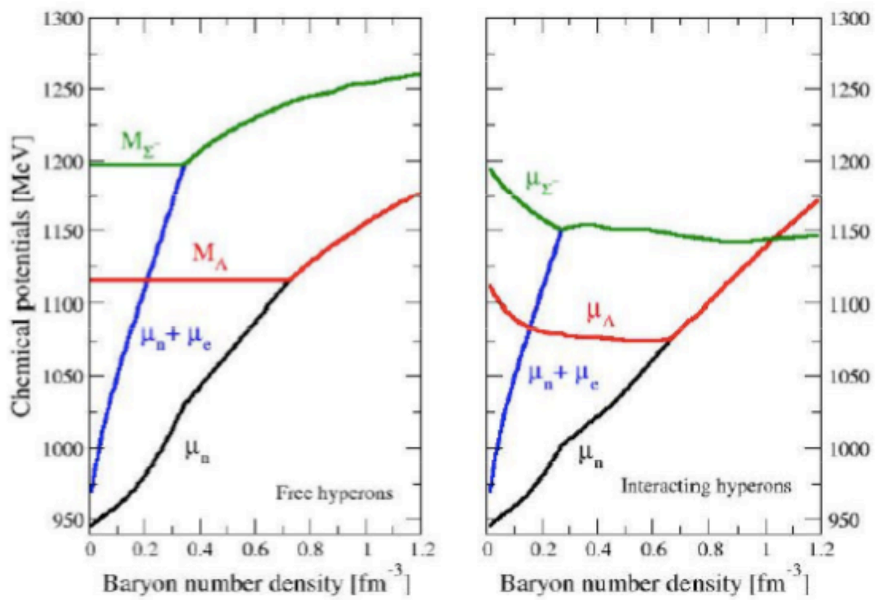
Ambartsumyan, Saakyan, 1960

"The core of a neutron star is a fluid of neutron rich matter in equilibrium with respect to the weak interactions (β stable matter)"

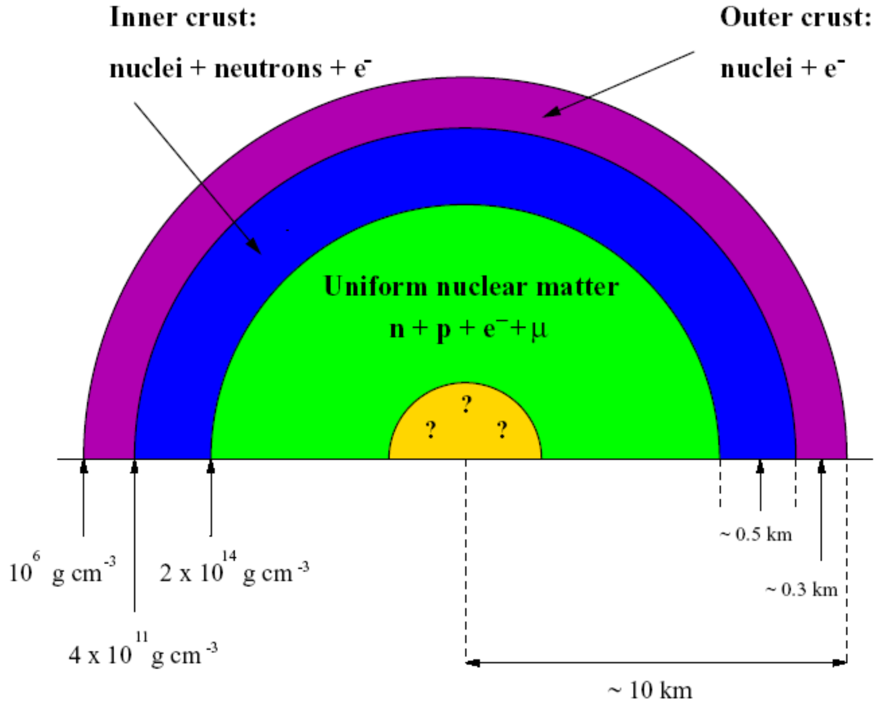
$$n \leftrightarrow p e^- \bar{\nu}_e \qquad (\mu_n = \mu_p + \mu_{e^-})$$

$$nn \leftrightarrow p\Sigma^- \quad \text{or} \quad e^- n \leftrightarrow \Sigma^- \nu_e \qquad (\mu_{e^-} + \mu_n = \mu_{\Sigma^-})$$

$$nn \leftrightarrow n\Lambda \quad \text{or} \quad e^- p \leftrightarrow \Lambda \nu_e \qquad (\mu_n = \mu_\Lambda)$$



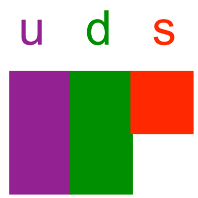
Thesis Isaac Vidaña, 2001



Strangeness

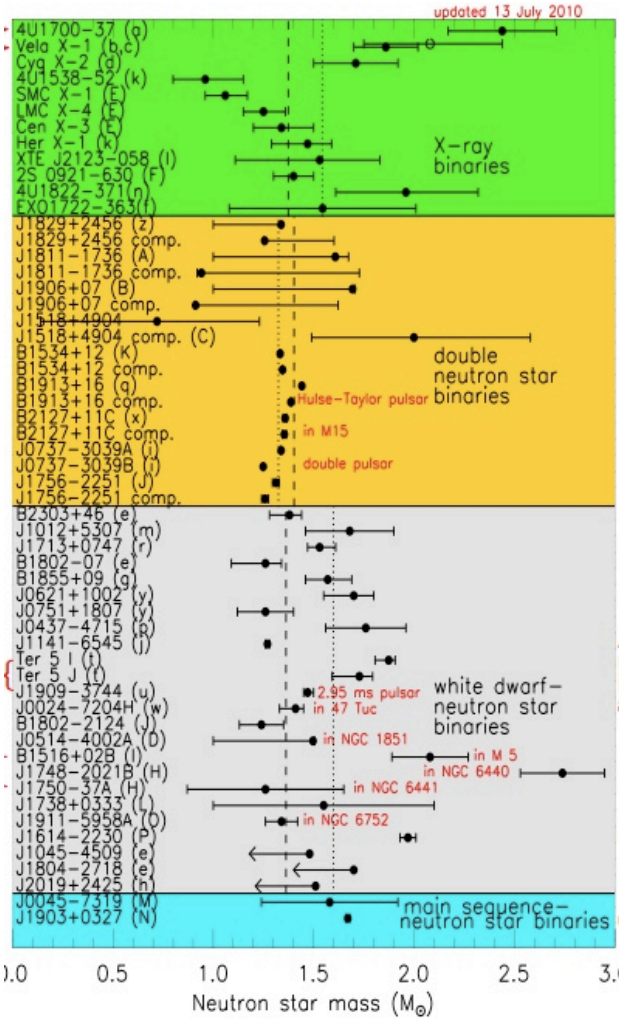
Confined

Σ, Λ, K

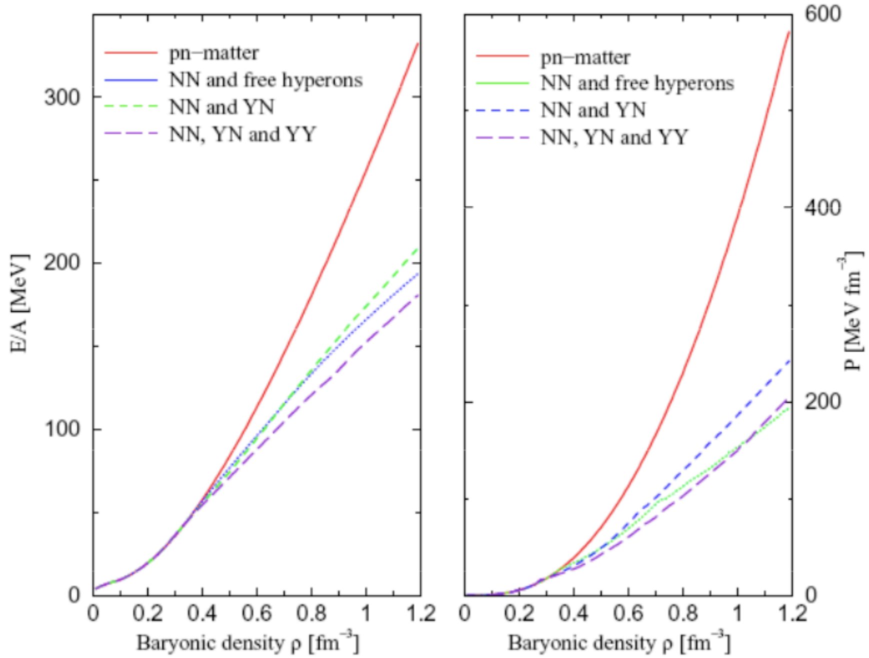


The composition of a neutron star depends on the hyperon properties in the medium (i.e. on the YN and YY interactions)

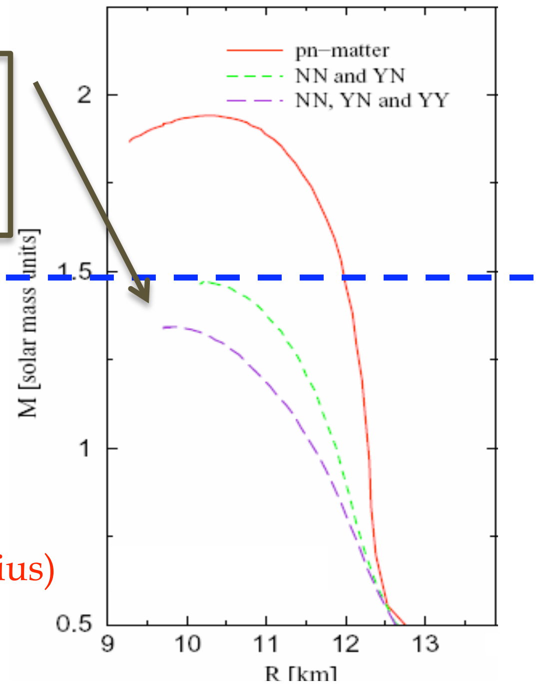
© J.M. Lattimer



Need for extra pressure at high density:
Improved YN, YY two-body interaction
Three-body forces: NNY, NYY, YYY



Microscopic EOS for hyperonic matter are "too" soft!



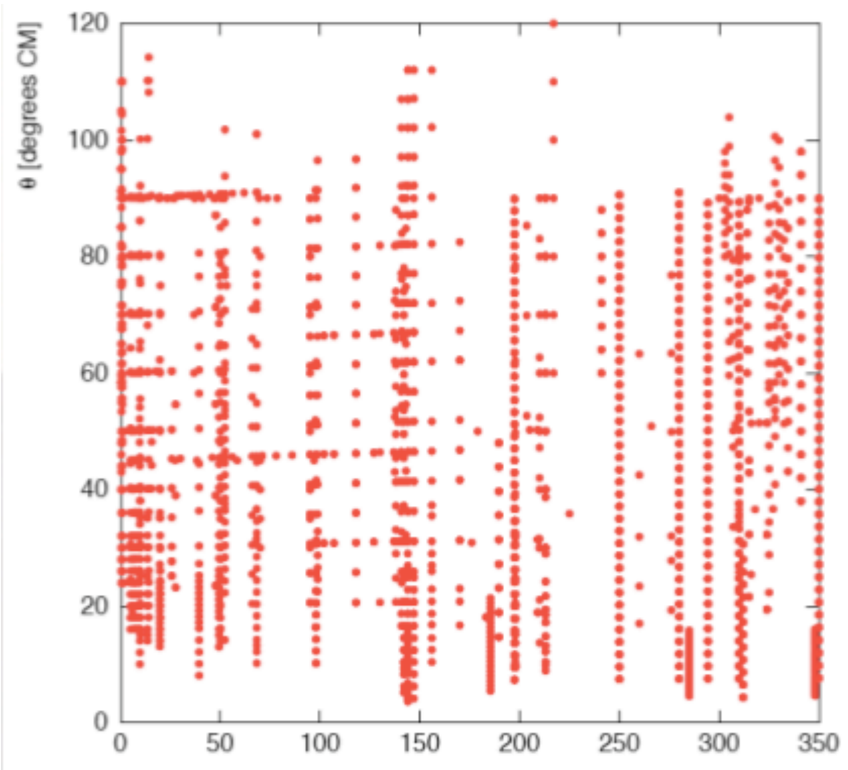
Influence of hyperons:
lower maximum masses
higher central densities
more compact (smaller radius)

Schulze, Polls, Ramos, Vidaña, PRC73, 058801 (2006)

How well do we know these (strong) interactions among hadrons?

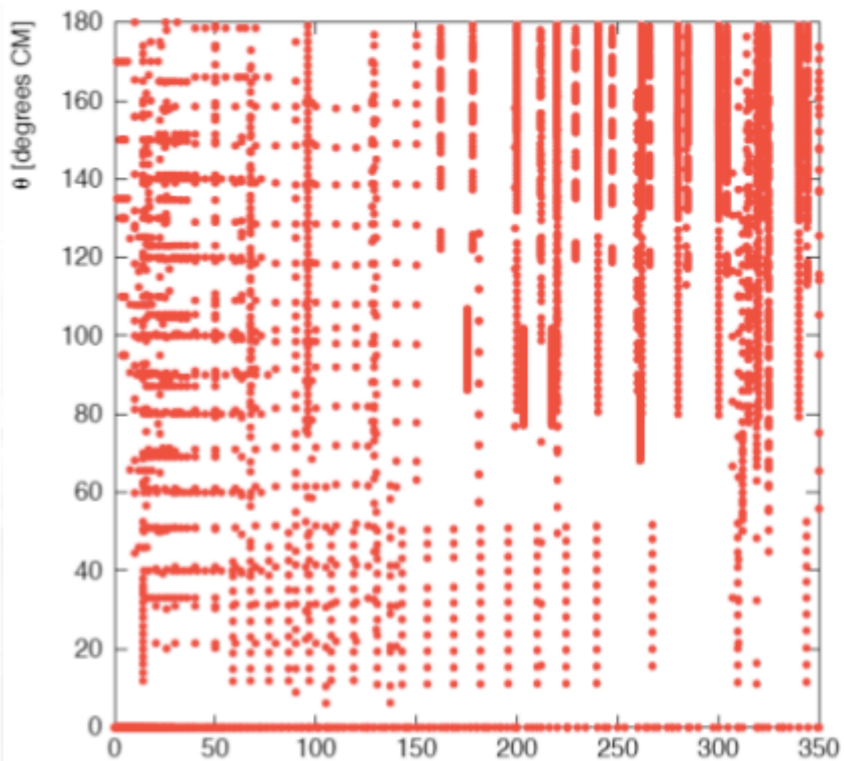
NN abundance plot

proton-proton



T_{lab} (MeV)

neutron-proton



T_{lab} (MeV)

Acknowledges go to Rob Timmermans

The low-energy YN "database"

~ 35 data points (many pre-1971) with large errors

Λp # = 12 6.5 MeV < T_{lab} < 50 MeV

$\Sigma^- p \rightarrow \Sigma^- p$ # = 6

Λn # = 6 9 MeV < Tlab < 12 MeV

$\Sigma^0 n$ # = 6

$\Sigma^+ p$ # = 4 9 MeV < Tlab < 13 MeV

+ 3 data from KEK-E289

"Ratio at rest" (inelastic capture ratio) of stopped Σ^- by protons:

$$r_R = \# \Sigma^0 / (\#\Sigma^0 + \#\Lambda) = 0.468(10)$$

Some differential cross sections of low quality

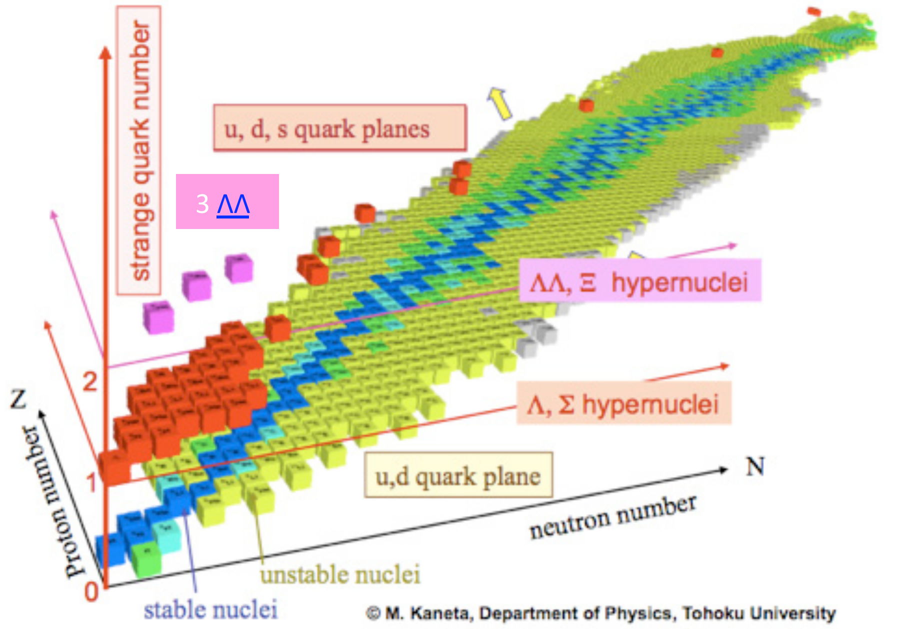
Additional information:

YN → Light hypernuclei:

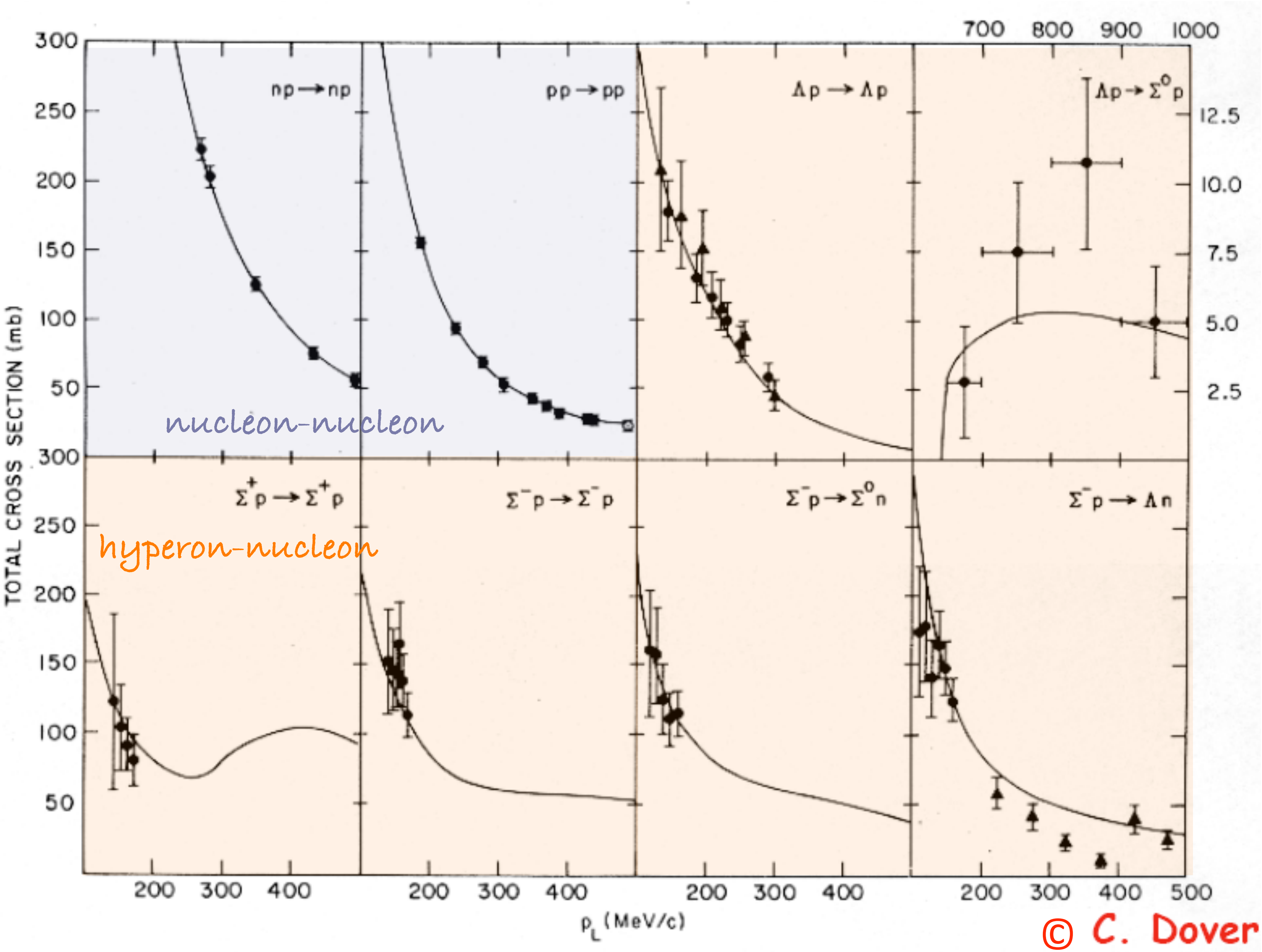
$^3H_\Lambda$, $^4He_\Lambda$, $^4H_\Lambda$, $^5He_\Lambda$

YY → $^6He_{\Lambda\Lambda}$, $^{10}Be_{\Lambda\Lambda}$, $^{13}B_{\Lambda\Lambda}$ …

39 Λ
1 Σ



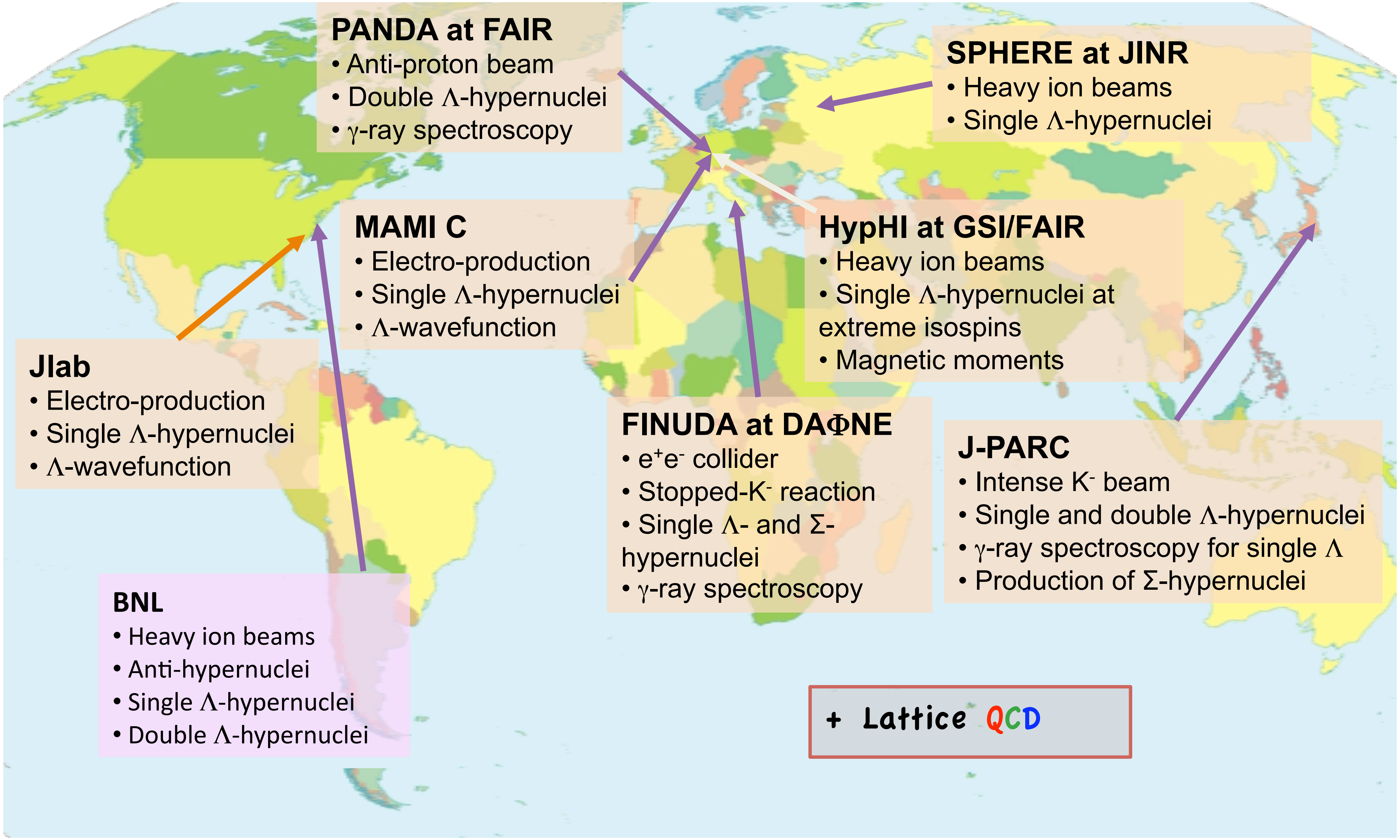
scattering events…



"strange" experimental program

J. Pochodzalla, IJMP E, Vol 16, no. 3 (2007) 925

very active field!



Now, suposse you can write down the effective theory for the low energy ΛN interaction, very low energy, below inelastic thresholds

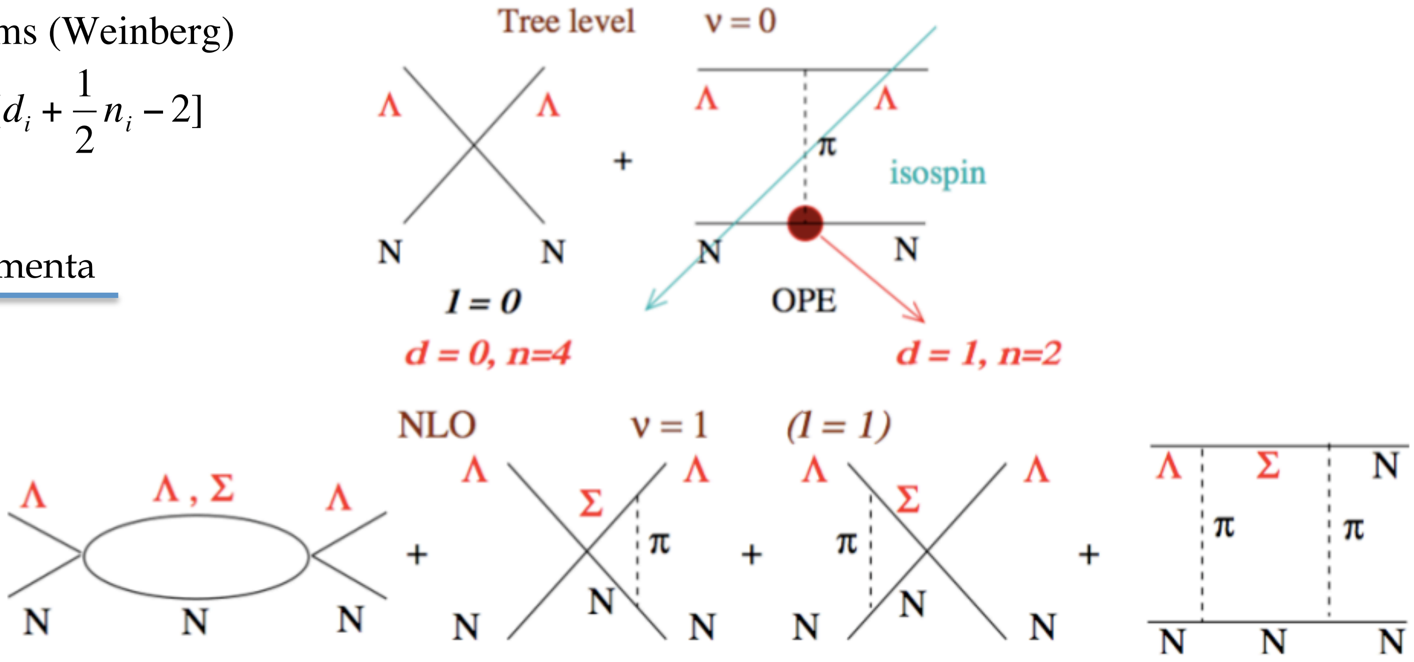
LO & NLO diagrams?

Now, suposse you can write down the effective theory for the low energy ΛN interaction

2 body diagrams (Weinberg)

$$\nu = 2L + \sum_i V_i[d_i + \frac{1}{2}n_i - 2]$$

power of momenta



$$L^{(LO)} =_{\Lambda\Lambda} C_0^{(^1S_0)}(\Lambda^T P^{(^1S_0)}N)\,(\Lambda^T P^{(^1S_0)}N) +_{\Lambda\Lambda} C_0^{(^3S_1)}(\Lambda^T P^{(^3S_1)}N)\,(\Lambda^T P^{(^3S_1)}N)$$

$$L^{(NLO)} =_{\Sigma\Lambda} C_0^{(^1S_0)}(\Sigma^T (i\tau_2)P^{(^1S_0)}N)\,(\Lambda^T (i\tau_2)P^{(^1S_0)}N) +_{\Sigma\Lambda} C_0^{(^3S_1)}(\Sigma^T (i\tau_2)P^{(^3S_1)}N)\,(\Lambda^T (i\tau_2)P^{(^3S_1)}N)$$

with $\quad P^{(^1S_0)} = \left(\frac{1}{\sqrt{2}}\right) i\sigma_2 \quad P^{(^3S_1)} = \left(\frac{1}{\sqrt{2}}\right) i\sigma_2\sigma^a$

Now, suposse you can write down the effective theory for the low energy ΛN interaction

Write the low-energy scattering parameters (scattering length, effective range) in terms of the coefficients of the Effective Theory

Beane, Bedaque, Parreño, Savage, Nucl. Phys. A747, 55-74 (2005); nucl-th/0311027

For example, for the singlet channel

$$a^{({}^1S_0)} = -\frac{\mu_{\Lambda N}}{2\pi}\left[\boxed{{}_{\Lambda\Lambda}C_0^{({}^1S_0)}} - \frac{3}{4\pi}\left(\boxed{{}_{\Sigma\Lambda}C_0^{({}^1S_0)}}\right)^2 \mu_{\Lambda N}\,\eta + {}_{\Sigma\Lambda}C_0^{({}^1S_0)}\,\frac{3 g_{\Sigma\Lambda} g_A \mu_{\Lambda N}}{2\pi f^2}\,\frac{\eta^2+\eta\, m_\pi + m_\pi^2}{\eta + m_\pi} - \frac{3 g_{\Sigma\Lambda}^2 g_A^2 \mu_{\Lambda N}}{4\pi f^4}\,\frac{2\eta^3 + 4\eta^2 m_\pi + 6\eta m_\pi^2 + 3 m_\pi^3}{2(\eta+m_\pi)^2}\right]$$

Extract LECs

$$r^{({}^1S_0)} = -\frac{1}{\mu_{\Lambda N}\,\pi}\left[\frac{2\pi}{\boxed{{}_{\Lambda\Lambda}C_0^{({}^1S_0)}}}\right]^2\left[\frac{3}{8\pi}\left(\boxed{{}_{\Sigma\Lambda}C_0^{({}^1S_0)}}\right)^2\frac{\mu_{\Lambda N}}{\eta} + {}_{\Sigma\Lambda}C_0^{({}^1S_0)}\,\frac{3 g_{\Sigma\Lambda} g_A \mu_{\Lambda N}}{2\pi f^2}\,\frac{3\eta^2 + 9\eta\, m_\pi + 8 m_\pi^2}{6(\eta+m_\pi)^3} - \frac{3 g_{\Sigma\Lambda}^2 g_A^2 \mu_{\Lambda N}}{4\pi f^4}\,\frac{6\eta^3 + 23\eta^2 m_\pi + 28\eta m_\pi^2 + 7 m_\pi^3}{12(\eta+m_\pi)^4}\right]$$

Result of the LQCD simulation

Why we should use Lattice QCD for the study of hadronic processes in nuclear physics?

improve our understanding of low-energy QCD
understanding nuclear processes from the underlying theory of strong interactions
first principle calculation
uncertainties can be quantified

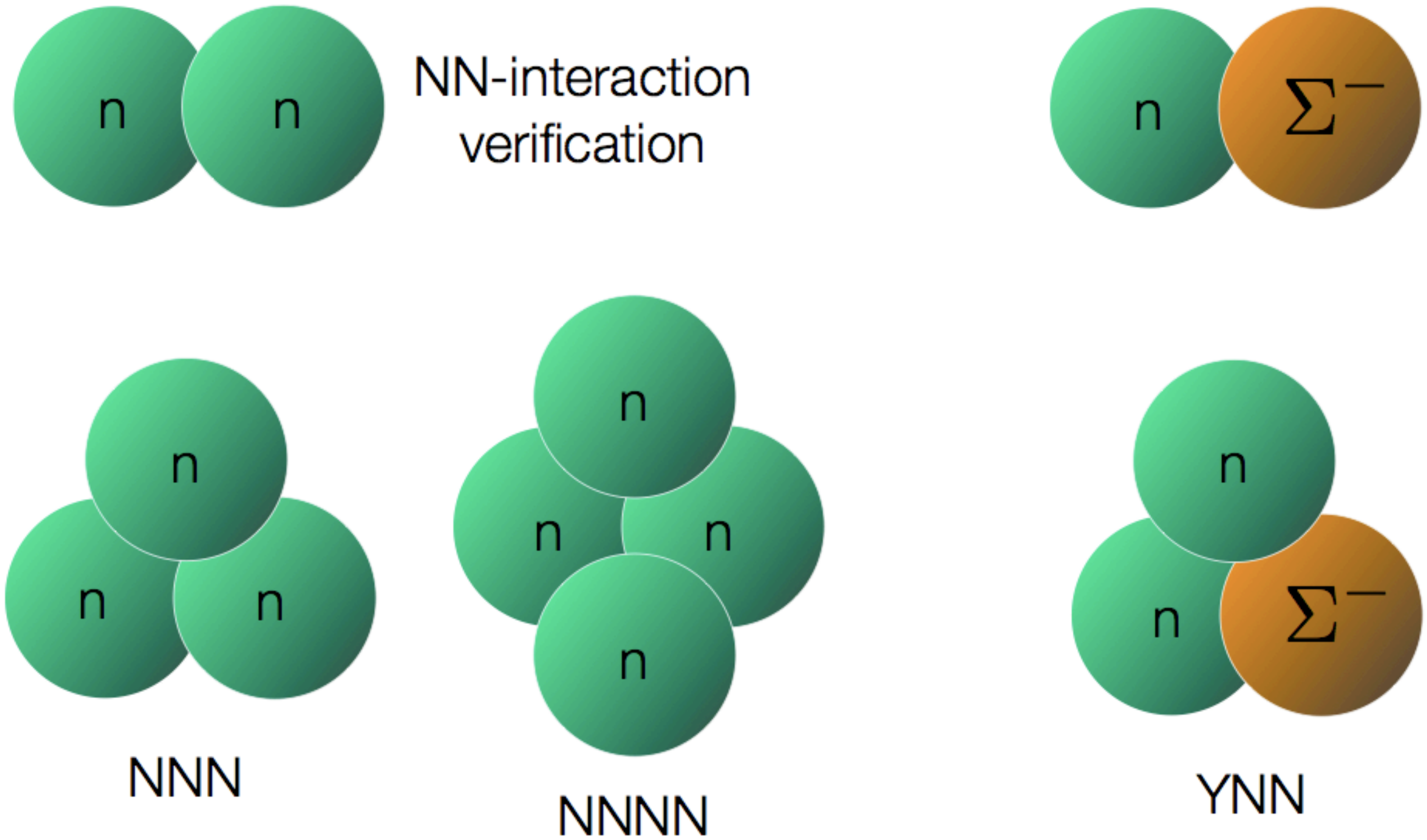
static properties of hadrons: masses

Focus on

study of the interactions among hadrons

LQCD calculations of hadronic interactions

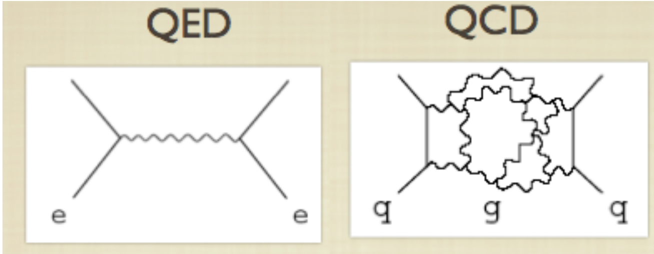
understanding nuclear processes from the underlying theory of the strong interactions



LQCD calculations
of hadronic interactions

- Some facts about QCD at low energies
- Overview of Lattice QCD basics
- Some results for hadronic observables in LQCD

QED Vs QCD



QED: $$L = \bar{\psi}(i\gamma^u \partial_u - m)\psi + \underbrace{e\bar{\psi}\gamma^u A_u \psi}_{\text{electron-}\gamma\text{ interaction}} - \frac{1}{4}F^{uv}F_{uv}$$

m=electron mass
ψ=electron spinor

A_u=photon field (1)
$F_{uv}=\partial_u A_v - \partial_v A_u$

QCD: $$L = \bar{q}_{jk}(i\gamma^u \partial_u - m)q_{jk} + \underbrace{g(\bar{q}_{jk}\gamma^u \lambda_a q_{jk})G^a_u}_{\text{quark-gluon interaction}} - \frac{1}{4}G^a_{uv}G^{uv}_a$$

gluon-gluon interaction

m=quark mass
j=color (1,2,3)
k=quark type (1-6)
q=quark spinor

G^a_u=gluon field (a=1-8)
$G^a_{uv}=\partial_u G^a_v - \partial_v G^a_u - g f_{abc} G^b_u G^c_v$

$[\lambda_a, \lambda_b] = i f_{abc} \lambda_c$

λ_a's (a=1-8) are the generators of SU(3).
λ_a's are 3x3 traceless hermitian matrices.

f_{abc} are real constants
$f_{abc} \equiv$ structure constants of the group

8 $SU_c(3)$ generators

$$\alpha\left(p^2\right) = \frac{\alpha(0)}{1 - X(p^2)}$$

For QCD:

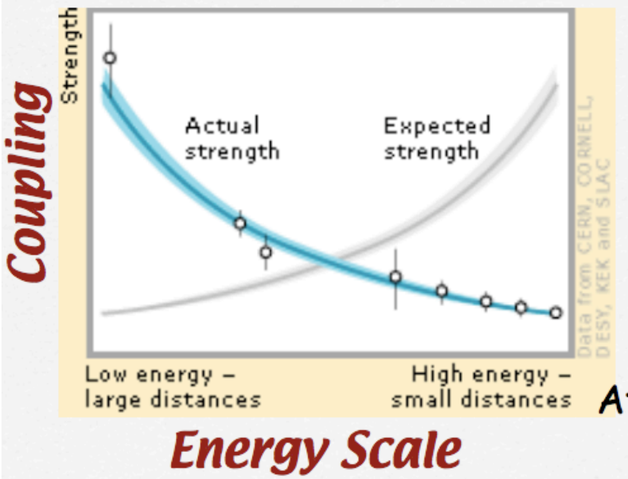
$$X\left(p^2\right) = \frac{\alpha_S\left(\mu^2\right)}{12\pi} \ln\left(\frac{p^2}{\mu^2}\right) \left[2N_f - 11N_c\right]$$

$$N_f = 6, \quad N_c = 3 \quad \Rightarrow \left[2N_f - 11N_c\right] < 0$$

and $\alpha\left(p^2\right)$ decreases as p^2 increases (small distances)

N_f = # flavors of quarks with mass < |p| / 2
μ = mass of the heaviest quark in the considered energy region
N_c = # de colors

Asymptotic Freedom



Gross, Politzer, Wilczek
2004 Nobel Prize

At low-energy, large distances, coupling becomes strong.
Perturbative series meaningless!
$g^2 \sim g^4$

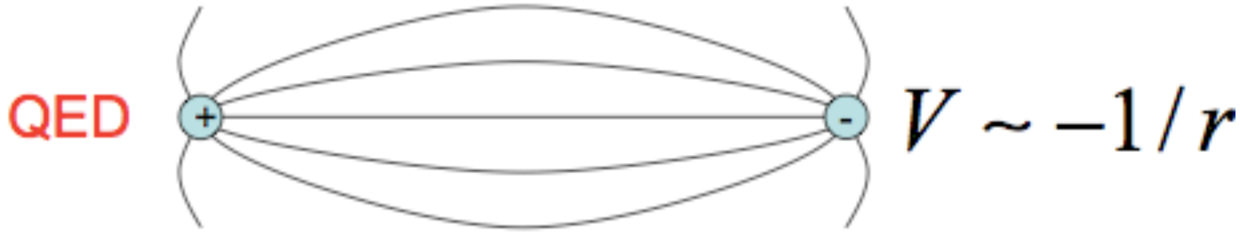
courtesy of Ross Young

The force between quarks decreases at short distances and increases when one tries to separate them

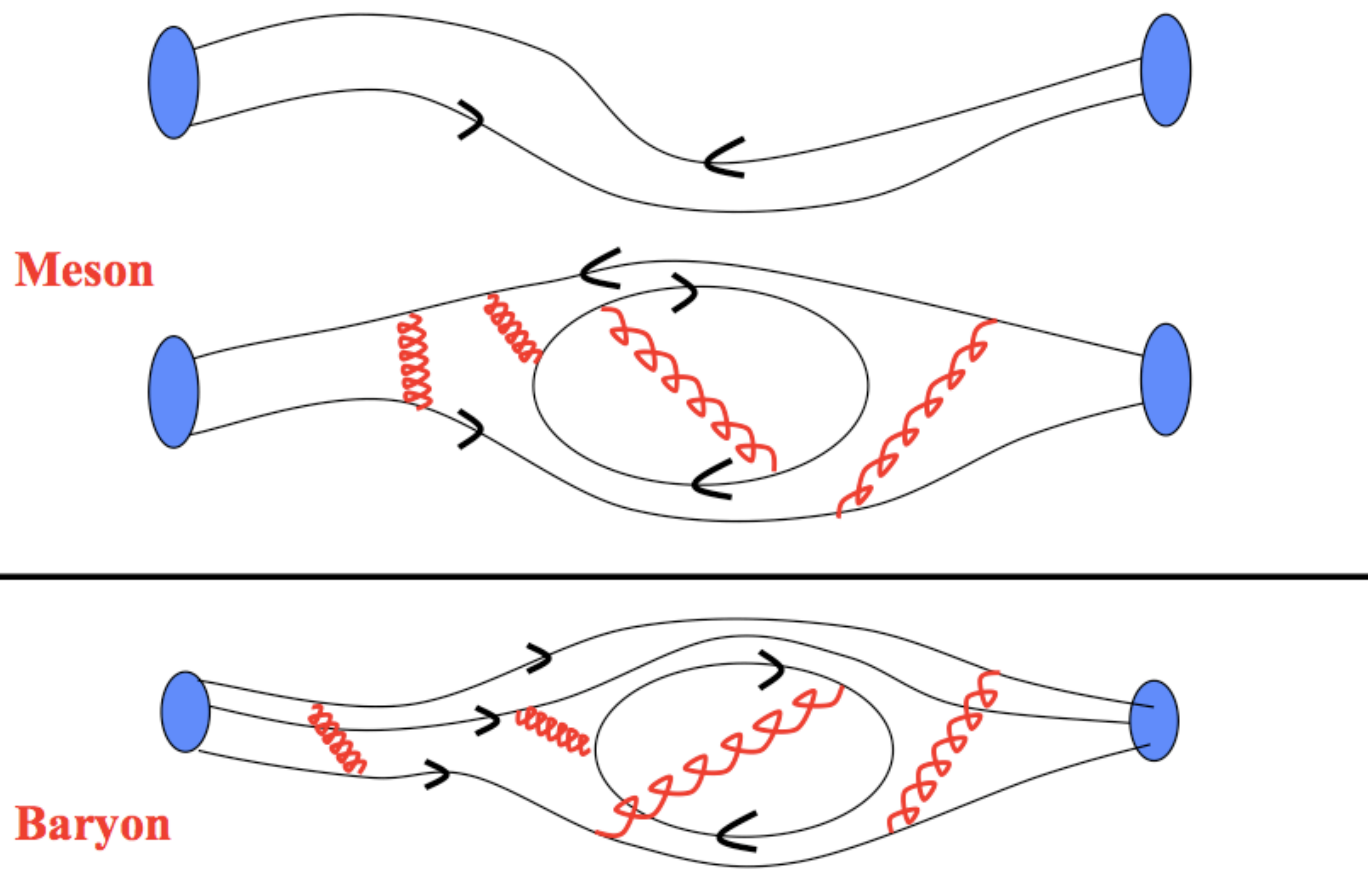
asymptotic freedom

confinement

At short distances (high energies) we have asymptotic freedom and the force between a quark and an anti-quark behaves as the one betweeen a e^- e^+ pair (QED)



At large distances (low energies ~ hadronic/nuclear physics) and as a consequence of the interaction between the gluons we obtain a potential that is linial with the distance and we have confinement.



QUARKS AND GLUONS ARE NOT SEEN AS ISOLATED STATES!

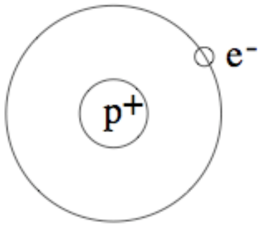
rajan@lanl.gov http://t8web.lanl.gov/people/rajan/ Masses

QCD vs QED. Bound states.

Bound states in QCD very different from QED

QED

Binding energy of a hydrogen atom
= sum of it constituent masses
(to a good approximation)
For nuclei: binding energy $\approx O(\text{MeV})$

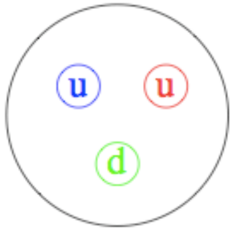


Hydrogen Atom

$M_e = 0.5$ MeV
$M_p = 938$ MeV
$E_{binding} = 13.6$ eV
(EM force)

QCD

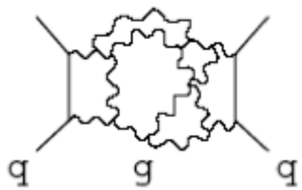
QCD



Proton

$M_u \sim 3$ MeV
$M_d \sim 6$ MeV
$M_p = 938$ MeV
(Strong force)

For the proton almost
all the mass is attributed
to the strong non-linear
interactions of the gluons.



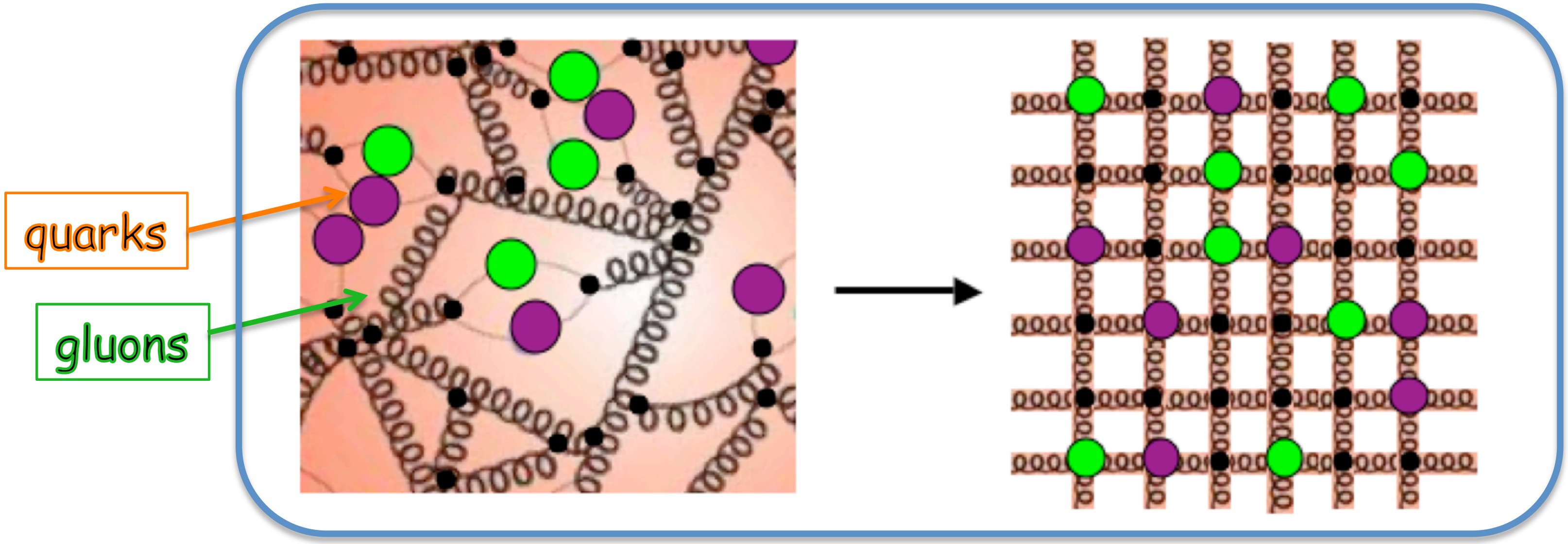
In principle, hadron masses and other hadronic observables can be calculated in QCD, although complicated...

Lattice QCD provides a well-defined approach to calculate observables non-perturbatively, starting from the QCD Lagrangian.

One can simulate the theory on a computer, using methods analogous to the ones used in Statistical Mechanics.

These simulations allow us to calculate correlation functions of hadronic operators and matrix elements of any operator between hadronic states in terms of the fundamental quark and gluon degrees of freedom.

Lattice Quantum Chromo Dynamics



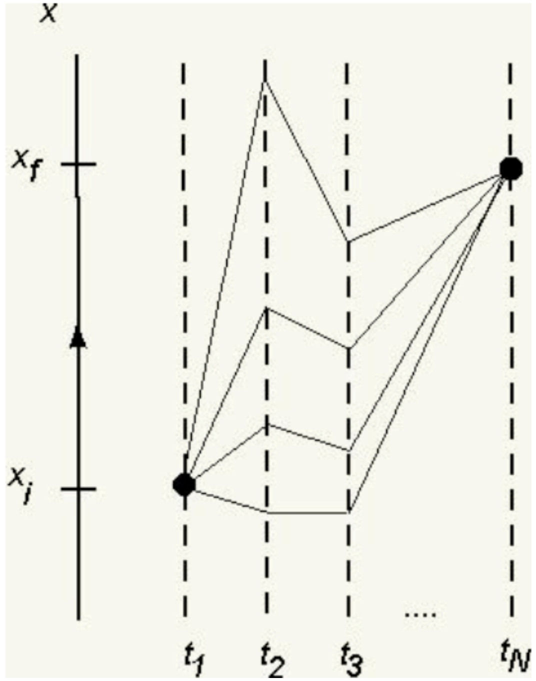
Courtesy of Dr. Thomas Luu (LLNLab)

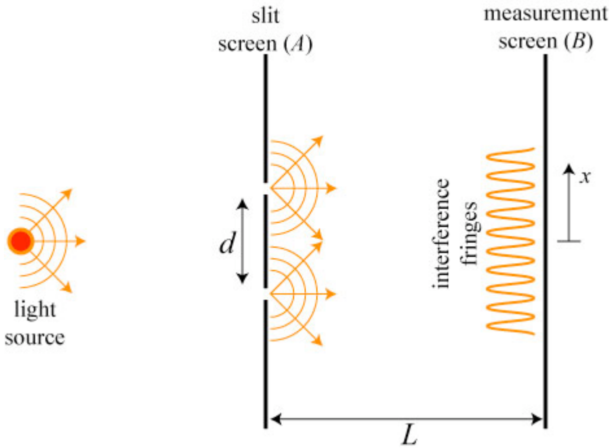
For numerical calculations in QCD, the theory is formulated on a (Euclidean) space-time lattice, as depicted in the right cartoon.

LQCD is a non-perturbative implementation of Field Theory, which uses the Feynman path-integral approach to evaluate transition matrix elements

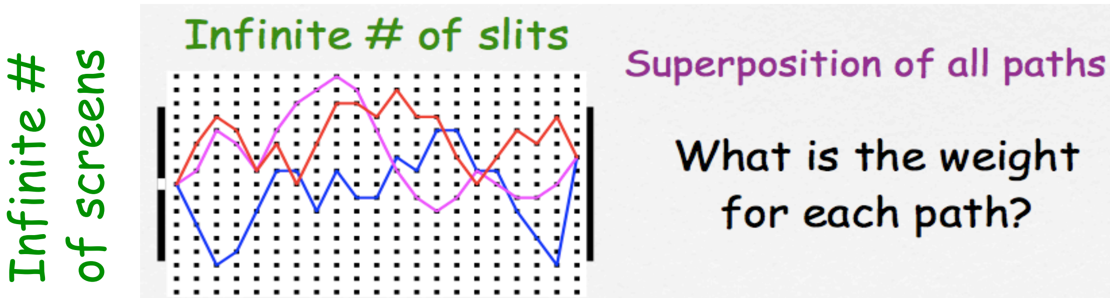
It replaces the classical notion of a single, unique trajectory for a system with a sum, or functional integral, over an infinity of possible trajectories to compute a quantum amplitude.

Study of the temporal evolution of particle states and of their interactions.





Young's double slit experiment

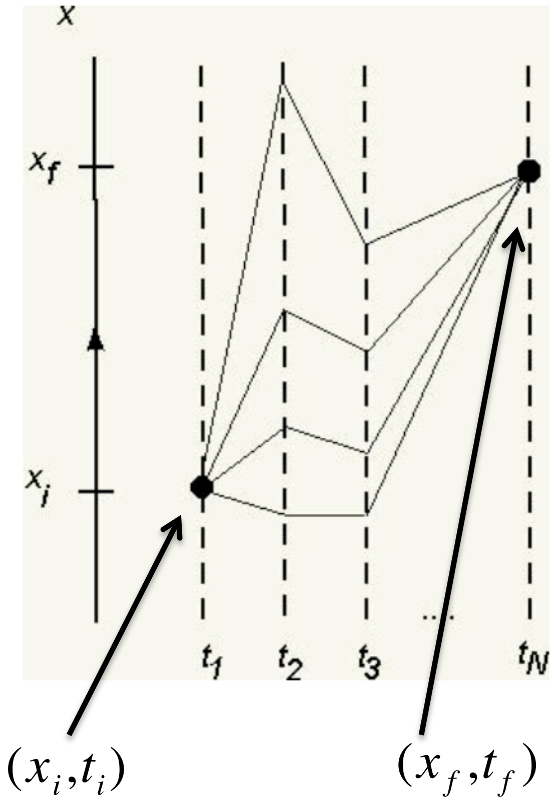


courtesy of Ross Young, Adelaide University

Path integrals in one dimension QM

Study of the temporal evolution of particle states and of their interactions.

Consider the time evolution of a quantum mechanical system: $\left|\psi(t_f)\right\rangle = e^{-iH(t_f-t_i)}\left|\psi(t_i)\right\rangle$



Evolution of a quantum state (x_i,t_i) to (x_f,t_f) :

$$\left\langle x_f,t_f \middle| x_i,t_i \right\rangle = \left\langle x_f \middle| e^{-iH(t_f-t_i)} \middle| x_i \right\rangle$$

Divide $t \equiv t_f - t_i$ into a large number N *of* intervals, of size $\Delta\,\mathrm{t} = t/N$ and insert $\hat{1} = \int dx_i \left|x_i\right\rangle\left\langle x_i\right| \quad i = 1,2,\ldots.\ N-1$

$$e^{-iHt} = e^{-iH\Delta t}\int dx_{N-1}\left|x_{N-1}\right\rangle\left\langle x_{N-1}\right|\ e^{-iH\Delta t}\int dx_{N-2}\left|x_{N-2}\right\rangle\left\langle x_{N-2}\right|\ \ldots\ e^{-iH\Delta t}\int dx_1\left|x_1\right\rangle\left\langle x_1\right| e^{-iH\Delta t}$$

For an **interacting particle** with $\hat{H} = \hat{H}_0 + \hat{V}(x) = \dfrac{\hat{p}^2}{2m} + V(\hat{x})$

$$\left\langle x_{k+1}\middle| e^{-i\Delta t\left(\frac{\hat{p}^2}{2m}+V(\hat{x})\right)} \middle| x_k\right\rangle \xrightarrow{\Delta t \to 0} \int \frac{dp}{2\pi}\left\langle x_{k+1}\middle|p\right\rangle e^{-i\Delta t\frac{p^2}{2m}}\, e^{-i\Delta t V(x_k)}\left\langle p\middle|x_k\right\rangle$$

$$\langle x_{k+1}|e^{-i\Delta t\left(\frac{\hat{p}^2}{2m}+V(\hat{x})\right)}|x_k\rangle \xrightarrow{\Delta t\to 0} \int\frac{dp}{2\pi}\langle x_{k+1}|p\rangle e^{-i\Delta t\frac{p^2}{2m}}e^{-i\Delta t V(x_k)}\langle p|x_k\rangle$$

$$\sim\int\frac{dp}{2\pi}e^{ip(x_{k+1}-x_k)}e^{-i\Delta t\frac{p^2}{2m}}e^{-i\Delta t V(x_k)} = \sqrt{\frac{2m\pi}{\Delta t}}e^{i\Delta t\sum_k\left[\frac{m}{2}\left(\frac{x_{k+1}-x_k}{\Delta t}\right)^2-V(x_k)\right]} + O(\Delta t^2)$$

$$\langle x_f,t_f|x_i,t_i\rangle = \langle x_f|e^{-iH(t_f-t_i)}|x_i\rangle = \int dx_{N-1}\int dx_{N-2}\cdots\int dx_1\, e^{i\Delta t\sum_k\left[\frac{m}{2}\left(\frac{x_{k+1}-x_k}{\Delta t}\right)^2-V(x_k)\right]}$$

$$\to \int_{x(0)=x_i}^{x(t)=x_f} D[x(t)]e^{iS_{classical}[x(t)]}$$

Lagrangian

And the evolution operator is the sum over all paths weighted by the exponential of the classical action.

The quantum propagation is expressed as a weighted sum over paths. The weight is a complex phase factor given by the exponential of i times the action *S*.

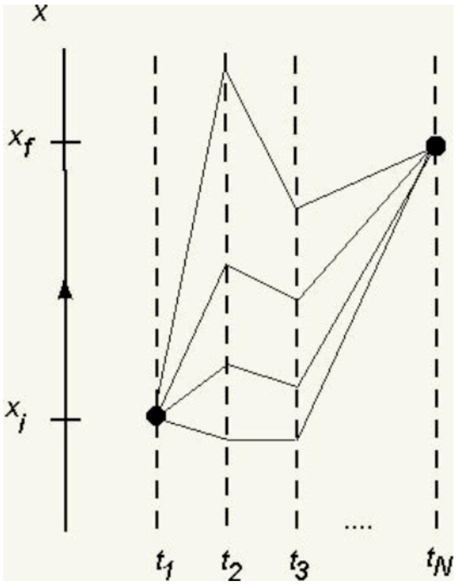
$$\langle x_{k+1}| e^{-i\Delta t\left(\frac{\hat{p}^2}{2m}+V(\hat{x})\right)}|x_k\rangle \xrightarrow{\Delta t\to 0} \int \frac{dp}{2\pi}\langle x_{k+1}|p\rangle e^{-i\Delta t\frac{p^2}{2m}} e^{-i\Delta t V(x_k)}\langle p|x_k\rangle$$

$$\sim \int \frac{dp}{2\pi} e^{ip(x_{k+1}-x_k)} e^{-i\Delta t \frac{p^2}{2m}} e^{-i\Delta t V(x_k)} = \sqrt{\frac{2m\pi}{\Delta t}}\, e^{i\Delta t\sum_k\left[\frac{m}{2}\left(\frac{x_{k+1}-x_k}{\Delta t}\right)^2 - V(x_k)\right]} + O(\Delta t^2)$$

$$\langle x_f,t_f|x_i,t_i\rangle = \langle x_f|e^{-iH(t_f-t_i)}|x_i\rangle = \int dx_{N-1}\int dx_{N-2}\cdots\int dx_1\, e^{i\Delta t\sum_k\left[\frac{m}{2}\left(\frac{x_{k+1}-x_k}{\Delta t}\right)^2 - V(x_k)\right]}$$

$$\to \int_{x(0)=x_i}^{x(t)=x_f} D[x(t)]\, e^{iS_{classical}[x(t)]}$$

Lagrangian



RULE

each path contributes a phase given by the classical action

$$A_i \propto \exp\left(i\int_i^f dt\, L(q(t))\right)$$

PATH INTEGRAL
Feynman, 1948

$$A = \int D(q)\, \exp\left(i\int_i^f dt\, L(q(t))\right)$$

$$\left\langle x_{k+1}\right| e^{-i\Delta t\left(\frac{\hat{p}^2}{2m}+V(\hat{x})\right)}\left|x_k\right\rangle \xrightarrow{\Delta t\to 0} \int \frac{dp}{2\pi}\left\langle x_{k+1}|p\right\rangle e^{-i\Delta t\frac{p^2}{2m}} e^{-i\Delta t V(x_k)}\left\langle p|x_k\right\rangle \qquad (1)$$

$$\sim \int \frac{dp}{2\pi} e^{ip(x_{k+1}-x_k)} e^{-i\Delta t\frac{p^2}{2m}} e^{-i\Delta t V(x_k)} = \sqrt{\frac{2m\pi}{\Delta t}}\, e^{i\Delta t \sum_k \left[\frac{m}{2}\left(\frac{x_{k+1}-x_k}{\Delta t}\right)^2 - V(x_k)\right]} + O(\Delta t^2)$$

$$\left\langle x_f, t_f | x_i, t_i\right\rangle = \left\langle x_f\right| e^{-iH(t_f-t_i)}\left|x_i\right\rangle = \int dx_{N-1}\int dx_{N-2}\cdots\int dx_1\, e^{i\Delta t \sum_k \left[\frac{m}{2}\left(\frac{x_{k+1}-x_k}{\Delta t}\right)^2 - V(x_k)\right]}$$

$$\to \int_{x(0)=x_i}^{x(t)=x_f} D[x(t)]\, e^{iS_{classical}[x(t)]}$$

Lagrangian

By rotating to Euclidean time in (1):

$$x_0 \equiv t \to -ix_4 \equiv -i\tau$$
$$p_0 \equiv E \to ip_4$$

$$x_E^2 = \sum_{i=1}^{4} x_i^2 = \vec{x}^2 - t^2 = -x_M^2$$
$$p_E^2 = \sum_{i=1}^{4} p_i^2 = \vec{p}^2 - E^2 = -p_M^2$$

$$e^{-\tau H} \to \int_{x(0)=x_i}^{x(t)=x_f} D[x_1, x_2, \ldots, x_{N-1}]\, e^{-\Delta\tau \sum_k \left[\frac{m}{2}\left(\frac{x_{k+1}-x_k}{\Delta t}\right)^2 + V(x_k)\right]}$$

Hamiltonian

The propagation amplitude is re-expressed in terms of the Euclidean action, S_E

$$e^{-\tau H} \rightarrow \int_{x(0)=x_i}^{x(t)=x_f} D\left[x_1, x_2, \ldots, x_{N-1}\right] e^{-\Delta\tau \sum_k \left[\frac{m}{2}\left(\frac{x_{k+1}-x_k}{\Delta t}\right)^2 + V(x_k)\right]}$$

basis of numerical simulations

Euclidean path real

The weight of each path is a real positive quantity, looking tlike a *Boltzmann factor*

Analogy with the partition function of a classical statistical mechanics system:

$$Z = Tr\left(e^{-\tau H}\right) = \int dx \left\langle x \middle| e^{-H\tau} \middle| x \right\rangle = \int D(x_1, x_2, \ldots, x_N)\, e^{-S(x_1, x_2, \ldots, x_N)}$$

$$S[x] \equiv \int_{\tau_i}^{\tau_f} d\tau\, L(x, \dot{x}) \equiv \int_{\tau_i}^{\tau_f} dt \left[\frac{m\, \dot{x}(\tau)^2}{2} + V\left(x(\tau)\right)\right]$$

$$\left\langle G[\phi] \right\rangle_T = \frac{\sum_\phi e^{-\frac{E[\phi]}{kT}} G[\phi]}{\sum_\phi e^{-\frac{E[\phi]}{kT}}}$$

Starting point is the QCD partition function in Euclidean space-time

$$Z = \int DA_\mu\, D\overline{\psi}\, D\psi\, \exp(-\underbrace{S_{QCD}}) = \int DA_\mu\, D\overline{\psi}\, D\psi\, \exp\left(-\int d^4x \left(\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \sum_f \overline{\psi}_f [D_\mu \gamma_\mu + m] \psi_f\right)\right)$$

real, positive weight → PROBABILITY

gluons (8)

anti-quarks (6)

quarks (6)

espin (4) x color (3)

$$\langle O \rangle = \frac{1}{Z} \int DA_\mu\, O\, e^{-S}$$

$$DA_\mu \equiv \prod_x dA_\mu(x)$$

functional integration

large # of degrees of freedom

dimension = 8 x 4 x 6 x 12 x 6 x 12 x # space points ~ 8 x 4 x 6 x 12 x 6 x 12 x 32^4 ~ 1.7 10^{11}

integrate the fermion fields (by gaussian integration)

$$Z = \int DA_\mu \det\left[M_f(A)\right] \exp(-S_{gluon}) = \int DA_\mu \exp(-S) \quad \text{with } S = \int d^4x \left(\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a\right) - \sum_{flavors} \log\left(\det\left[M_f(A)\right]\right)$$

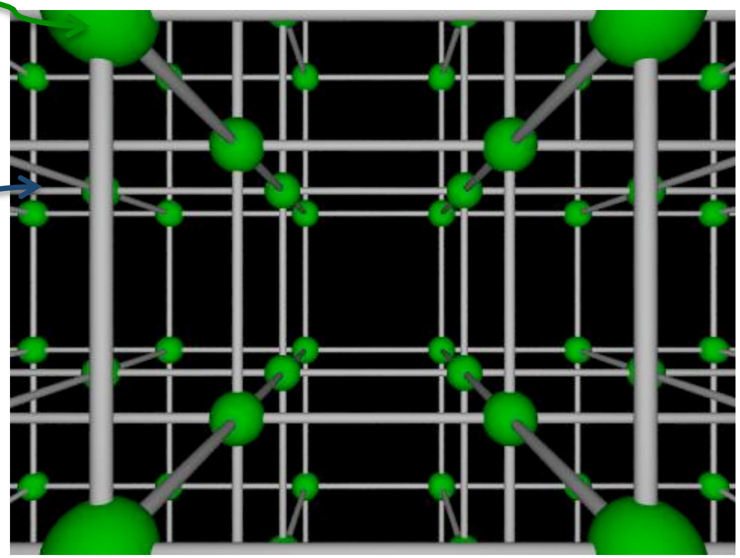
Lattice formulation: Discrete space-time & Use a discrete action

Evaluate a path ordered exponential between neighbor sites

periodic (anti-periodic) boundary conditions

Quarks

Gluons



space-time lattice

$N_s \times N_s \times N_s \times N_t$

$m_\pi L >> 2\pi$ $\qquad$ $b << 1 fm$

$\frac{1}{m_\pi} \approx 1.4\ fm$ $\qquad$ $1\ fm\ =\ 10^{-15} m$

$\langle \; | \; | \; \rangle \longrightarrow$ time-ordered correlation functions

The quark fields in *O* are re-expressed in terms of quark propagators using Wick's Theorem for contracting fields (removing the dependence of quarks as dynamical fields)

Basic building block for fermionic quantities:

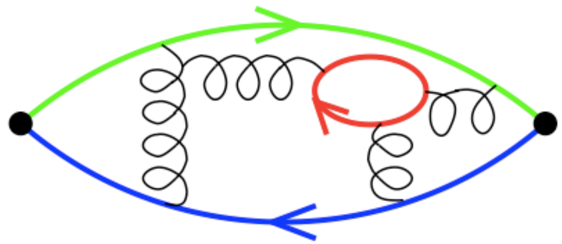
Feynman propagator $\qquad S_F(y,j,b;x,i,a) = \left(M^{-1}\right)^{y,j,b}_{x,i,a}$ $\qquad$ site, spin, color

inverse of the Dirac operator calculated on a given background field

$$Z = \int DA_\mu \det\left[M_f(A)\right] \exp(-S_{gluon}) = \int DA_\mu \exp(-S)$$

very demanding

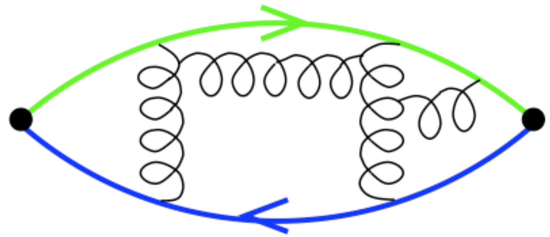
$$(S = S_{gluon} + S_f) \qquad S_f = \bar{\psi} M_f(A) \psi$$



(B) Full QCD

Nowadays many simulations are dynamical, but until a few years ago there were:

quenched $\longrightarrow$ $\det\left[M_f(A)\right] = \text{constant}$ $\longleftarrow$



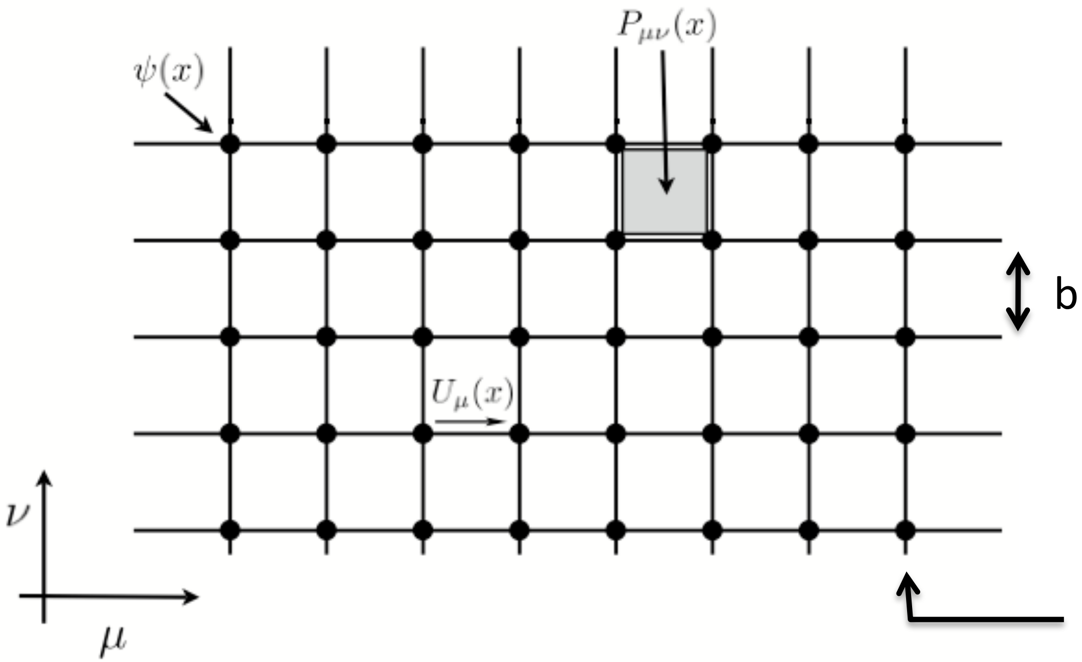
(A) Quenched QCD: quark loops neglected

partially quenched

$$\left\{ \begin{array}{ll} \text{in the determinant:} & \det\left[M_f(A)\right] \equiv \det\left(\not{D}[A] + m_{sea}\right) \\ \text{in the quark propagator:} & S[A] \equiv \left(\not{D}[A] + m_{val}\right)^{-1} \end{array} \right\} \quad \text{with } m_{sea} \neq m_{val}$$

one needs to develop effective theories that take into account this mixing in order to extrapolate to physical results

Define parallel transporters on links between neighboring sites in the lattice



$$U_\mu(x) \equiv e^{ig\int_x^{x+b\mu} A_\mu dx^\mu} \cong 1 + igbA_\mu(x + b\hat{\mu}/2)$$

$$U_{-\mu}(x) \equiv e^{ig\int_x^{x-b\mu} A_\mu dx^\mu} \cong U_\mu(x - b\hat{\mu}) \cong 1 - igbA_\mu(x - b\hat{\mu}/2)$$

SU(3) matrices, "links"

$q_\alpha^i(x)$ i: color index
α: spin index

Basic discretized operators:

- Ordinary derivative: $$\partial_\mu q(x) \equiv \frac{1}{2b}\left[q(x + b\hat{\mu}) - q(x - b\hat{\mu})\right]$$
- Covariant derivative: $$D_\mu q(x) \equiv \frac{1}{2b}\left[U_\mu(x)q(x + b\hat{\mu}) - U_{-\mu}(x)q(x - b\hat{\mu})\right]$$

Exercise: Check that in the continuum limit we recover the QCD covariant derivative

For an arbitrary functional $\Gamma(x)$, we write the weighted average over paths with weight $e^{-S[x]}$ as: $\langle\Gamma[x]\rangle = \dfrac{\int Dx(t)\Gamma[x]e^{-S[x]}}{\int Dx(t)e^{-S[x]}}$

- Produce N gauge field configurations {U} with probability distribution P(U) $\left(P[U] \propto e^{-S[U]}\right)$

  $\left\{U^{[i]}\right\}$, (Markov process)

  each configuration is created by the preceding one:

  $$\mathrm{P}\left(U^{[i-1]} \to U^{[i]}\right) P\left(U^{[i-1]}\right) = \mathrm{P}\left(U^{[i]} \to U^{[i-1]}\right) P\left(U^{[i]}\right)$$

Basic Monte Carlo algorithm
(Metropolis, heatbath, …)

- Calculate expectation value for this distribution (Monte Carlo estimator)

  $$\langle\Gamma[U]\rangle \approx \overline{\Gamma} \equiv \frac{1}{N_{cf}} \sum^{N_{cf}} \Gamma[U]$$

  $$\langle 0|O(\ldots)|0\rangle^{latt} \cong \frac{1}{N} \sum_i O\left(\ldots\left[U^{[i]}\right]\right)$$

Average over the ensemble of configurations

- Results have statistical errors $\sim \dfrac{1}{\sqrt{N}}$

  number of uncorrelated gauge configurations

$$\sigma_{\overline{\Gamma}}^2 \approx \frac{1}{N_{cf}} \left\{ \frac{1}{N_{cf}} \sum_{\alpha=1}^{N_{cf}} \Gamma^2[x^{(\alpha)}] - \overline{\Gamma}^2 \right\} \xrightarrow{N_{cf} >>1} \frac{\langle\Gamma^2\rangle - \langle\Gamma\rangle^2}{N_{cf}}$$

(Monte Carlo uncertainty)

- Repeat this process very many times (changing randomly the location of the source propagator for instance) and average over the results

- Produce N gauge field configurations {U} with probability distribution P(U).
- Calculate expectation value for this distribution.
- Repeat this process very many times.
- Average over results.
- Results have statistical errors.

Basic algorithm
(Monte Carlo method)

Solve a linear system of equations: $D^{\dagger}(U)[m]\, D(U)[m]\, \chi = \phi$

→ Condition number ≈ 1/m

Extrapolations to connect with real life

source of systematic errors in lattice QCD simulations

$L \to \infty$ $\qquad$ $b \to 0$ $\qquad$ $m_q \to m_q^{physical}$

Published works have L ≤ 4 fm (~6 fm), b < ~ 0.1 fm and m_π ≥ 400 MeV (200 MeV)

Procedure $\quad$ Configurations ⟹ Compute propagators ⟹ Compute correlators

Systematic errors:

Finite volume

For enough large lattices , the finite volume corrections to the mass of a given state fall off as e^{-ML} (Lüscher'88)

Ex. take the pion $m_\pi L >\sim 2\pi \quad \Rightarrow \quad m_\pi \sim 140\, MeV \quad \Rightarrow \quad L \sim 9\, fm$

i.e. quantum mechanical properties of the pion (size ~1 fm) are unaltered if the box size is of the order of 9 fm (expensive!)

Due to the required periodic/anti-periodic boundary conditions, each observable computed in the lattice suffers from unphysical contribution from mirrow states.

Lattice spacing

Introduces discretization errors
One tries to reduce their size by improving the lattice action and the operators
The difference between the computed correlation function and their continuum limit is of the order of b (b^2 in improved lattice actions)
Present calculations typically have b ~ 0.1 fm

Light quark masses

Physical u and d masses are too light to simulate on current lattices
Usually m_u=m_d >> physical values and then, perform a chiral extrapolation

Hadron masses are two-point correlation functions

$$C(\Gamma^{\nu},\vec{p},t) = \sum_{\vec{x}_2} e^{-i\vec{p}\vec{x}_2}\Gamma^{\nu}\left\langle J(\vec{x}_2,t_2)\bar{J}(\vec{x}_0,t_0)\right\rangle$$

projects onto zero momentum

spin tensor

sink

source

interpolating operators

The validity of lattice QCD can be tested through the successful calculation of low-lying hadron masses

State	I^G	J^{PC}	Operator J
scalar	1^-	0^{++}	$\bar{q}q'$
	1^-	0^{++}	$\bar{q}\gamma^0 q'$
pseudoscalar	1^-	0^{-+}	$\bar{q}\gamma^5 q'$
	1^-	0^{-+}	$\bar{q}\gamma^0\gamma^5 q'$
vector	1^+	1^{--}	$\bar{q}\gamma^{\mu} q'$
	1^+	1^{--}	$\bar{q}\gamma^{\mu}\gamma^0 q'$
axial	1^-	1^{++}	$\bar{q}\gamma^{\mu}\gamma^5 q'$
tensor	1^+	1^{+-}	$\bar{q}\gamma^{\mu}\gamma^j q'$
octet	$\frac{1}{2}$	$\frac{1}{2}^-$	$(q^{Ti}\gamma^2\gamma^0 q'^j)(\gamma^5 q''^k)\varepsilon_{ijk}$
	$\frac{1}{2}$	$\frac{1}{2}^-$	$(q^{Ti}\gamma^2\gamma^0\gamma^5 q'^j)(q''^k)\varepsilon_{ijk}$
decuplet	$\frac{3}{2}$	$\frac{3}{2}^+$	$(q^{Ti}\gamma^2\gamma^0\gamma^i q'^j)(q''^k)\varepsilon_{ijk}$

$$\Gamma_{\pm} = \frac{1}{2}(1 \pm \gamma_0)$$

positive/negative energy (parity) states

$$J^{\pi} = \left(\frac{1}{2}\right)^{+}$$

$$p_{\alpha}(\vec{x},t) = \varepsilon^{ijk}u_{\alpha}^{i}(\vec{x},t)\left(u_{\alpha}^{j^T}(\vec{x},t)C\gamma_5 d^k(\vec{x},t)\right)$$

$$\Lambda_{\alpha}(\vec{x},t) = \varepsilon^{ijk}s_{\alpha}^{i}(\vec{x},t)\left(u_{\alpha}^{j^T}(\vec{x},t)C\gamma_5 d^k(\vec{x},t)\right)$$

$$\Sigma_{\alpha}^{+}(\vec{x},t) = \varepsilon^{ijk}u_{\alpha}^{i}(\vec{x},t)\left(u_{\alpha}^{j^T}(\vec{x},t)C\gamma_5 s^k(\vec{x},t)\right)$$

$$\Xi_{\alpha}^{0}(\vec{x},t) = \varepsilon^{ijk}s_{\alpha}^{i}(\vec{x},t)\left(u_{\alpha}^{j^T}(\vec{x},t)C\gamma_5 s^k(\vec{x},t)\right)$$

charge conjugation matrix

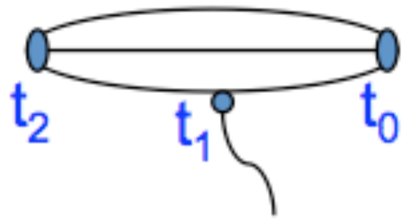
Two-point correlators

$$C(\Gamma^{\nu},\vec{p},t) = \sum_{\vec{x}_2} e^{-i\vec{p}\vec{x}_2}\Gamma^{\nu}\left\langle J(\vec{x}_2,t_2)\bar{J}(\vec{x}_0,t_0)\right\rangle$$

ex: masses

$$\sum_{\vec{x}_2} e^{-i\vec{p}_2\vec{x}_2}\left\langle J_{\pi}(\vec{x}_2,t_2)J^{\dagger}{}_{\pi}(\vec{x}_0,t_0)\right\rangle$$

Three-point correlators



$$C_{3\hat{O}}(\Gamma^{\nu};\vec{p}',\vec{p};,t_2,t_1) = \sum_{\vec{x}_2,\,\vec{x}_1} e^{-i\vec{p}'\vec{x}_2} e^{+i(\vec{p}'-\vec{p})\vec{x}_1}\Gamma^{\nu}\left\langle J(\vec{x}_2,t_2)\hat{O}(\vec{x}_1,t_1)\bar{J}(\vec{x}_0,t_0)\right\rangle$$

ex: pion form factor

$$\sum_{\vec{x}_2,\vec{x}} e^{-i\vec{p}_2\vec{x}_2}\, e^{i\vec{q}\vec{x}}\left\langle J_{\pi}(\vec{x}_2,t_2)\, V_{\mu}(\vec{x}_1,t_1)\, J_{\pi}^{+}(\vec{x}_0,t_0)\right\rangle$$

with q the momentum transfer $q = (p_2 - p_0)$

$$f_{\pi}(q^2) = 1 - \frac{1}{6}\left\langle r^2\right\rangle q^2 + O(q^4)$$

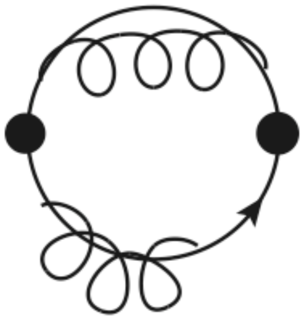
$$\rightarrow \quad \left\langle r^2\right\rangle = -6\left.\frac{df_{\pi}(q^2)}{dq^2}\right|$$

Example: *charged pion propagator* $\boxed{\pi^+ = \bar{d}\gamma_5 u}$

$$< \pi^\dagger (x)\pi(y) >=< \bar{u}(x)\gamma_5 d(x)\bar{d}(y)\gamma_5 u(y) >$$
$$= \int \mathcal{D}U \mathcal{D}u \mathcal{D}\bar{u} \mathcal{D}d \mathcal{D}\bar{d}[\bar{u}\gamma_5 d\bar{d}\gamma_5 u] e^{\int(-\bar{u}G_u^{-1}u - \bar{d}G_d^{-1}d) + S_G} / Z$$
$$= \int \mathcal{D}U \det[G_u] \det[G_d] G_u(x,y)\gamma_5 G_d(y,x)\gamma_5 e^{-S_G} / Z$$
$$\Leftrightarrow < \pi^\dagger (x)\pi(y) >=< \bar{u}(x)\gamma_5 d(x)\bar{d}(y)\gamma_5 u(y) > .$$

$$G_u(y,x) = u(x)\bar{u}(y)$$

Wick contractions
(equivalent to the path integral)



$$< \pi^+ (x)\pi(y) >=< Tr[G_u(x,y)\gamma_5 G_d(y,x)\gamma_5] >$$

(connected quark diagram)

Example: *neutral pion propagator*

$$\pi^0 = \frac{\bar{u}u + \bar{d}d}{\sqrt{2}}$$

$$G_u(y,x) = u(x)\bar{u}(y)$$

$$< \pi^\dagger(x)\pi(y) >=< \frac{1}{2}(\bar{u}(x)u(x) + \bar{d}(x)d(x))(\bar{u}(y)u(y) + \bar{d}(y)d(y)) >$$

$$= \frac{1}{2}[< \bar{u}(x)u(x)\bar{u}(y)u(y) > + < \bar{d}(x)d(x)\bar{d}(y)d(y) >$$

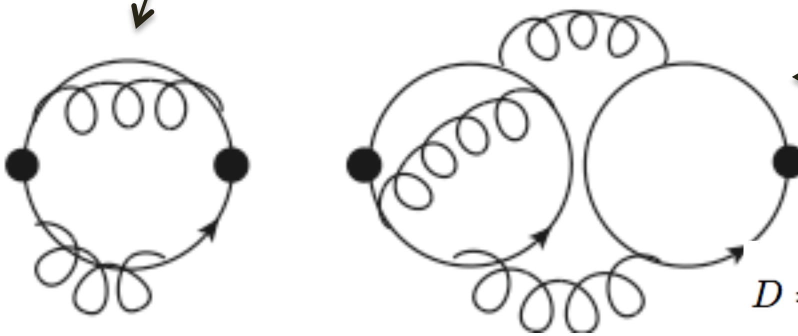
$$+ < \bar{d}(x)d(x)\bar{u}(y)u(y) > + < \bar{u}(x)u(x)\bar{d}(y)d(y) >]$$

$$= \frac{1}{2}[< \bar{u}(x)u(x)\bar{u}(y)u(y) > + < \bar{d}(x)d(x)\bar{d}(y)d(y) >$$

$$+ < \bar{u}(x)u(x)\bar{u}(y)u(y) > + < \bar{d}(x)d(x)\bar{d}(y)d(y) >$$

$$+ < \bar{d}(x)d(x)\bar{u}(y)u(y) > + < \bar{u}(x)u(x)\bar{d}(y)d(y) >] .$$

$$< \pi^+(x)\pi(y) >=< Tr[G_u(x,y)\gamma_5 G_d(y,x)\gamma_5] >$$



$$G_d(x,y) = \gamma_5 G_u^+(y,x)\gamma_5$$

$$D = \frac{1}{2} < Tr[G_u + G_u^+]Tr[G_u + G_u^+] >= 2 < (Tr[Re(G_u)])^2 >$$

(connected and disconnected quark diagrams)

Masses of (colourless) QCD bound states (locate the source at t=0)

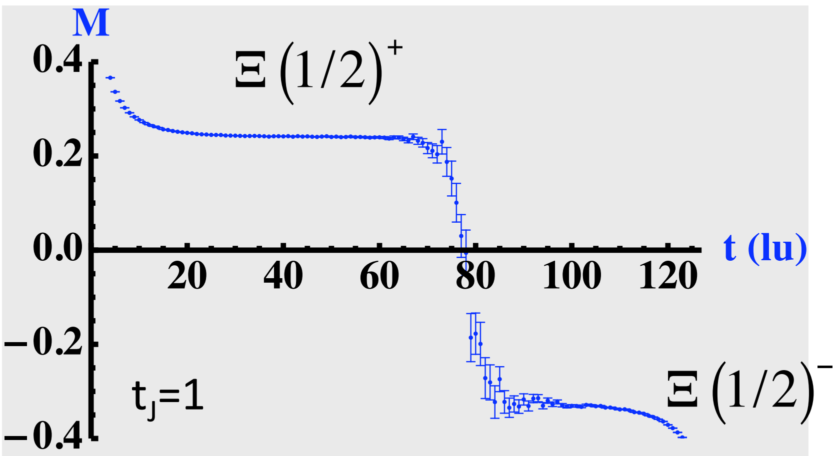
$$C(t) = \langle 0|\phi(t)\phi^{\dagger}(0)|0\rangle \xrightarrow[\phi(t)=e^{Ht}\phi e^{-Ht}]{} \langle\phi|e^{-Ht}|\phi\rangle$$

Insert a complete set of energy eigenstates:

mass

$$C(t) = \sum_n \langle\phi|e^{-Ht}|n\rangle\langle n|\phi\rangle = \sum_n |\langle\phi|n\rangle|^2 e^{-E_n t} \xrightarrow{t\to\infty} Z e^{-E_0 t}$$

i.e. one can obtain the energy of the state provided we see the large time exponential fall-off of the correlation function (Euclidean time evolution suppresses excited states)

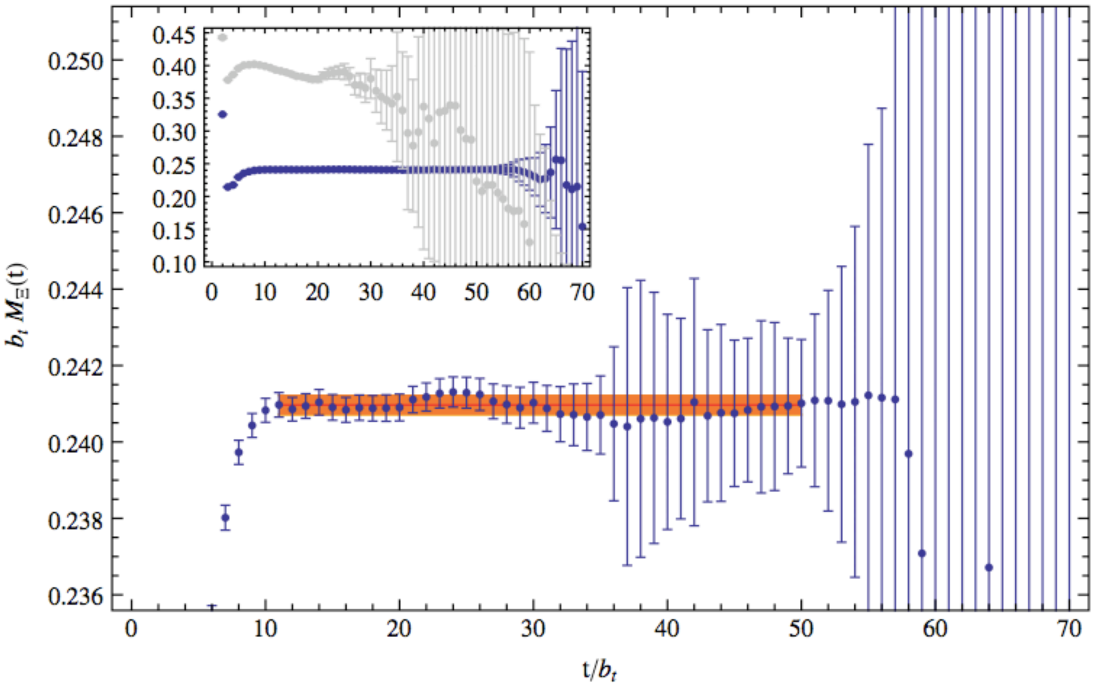


effective mass plot

$$\frac{1}{t_J}\log\frac{C_A(t)}{C_A(t+t_J)} = m_A$$

Tracking sub-leading exponential fall-offs can give us excited states

Tracking sub-leading exponential fall-offs can give us excited states

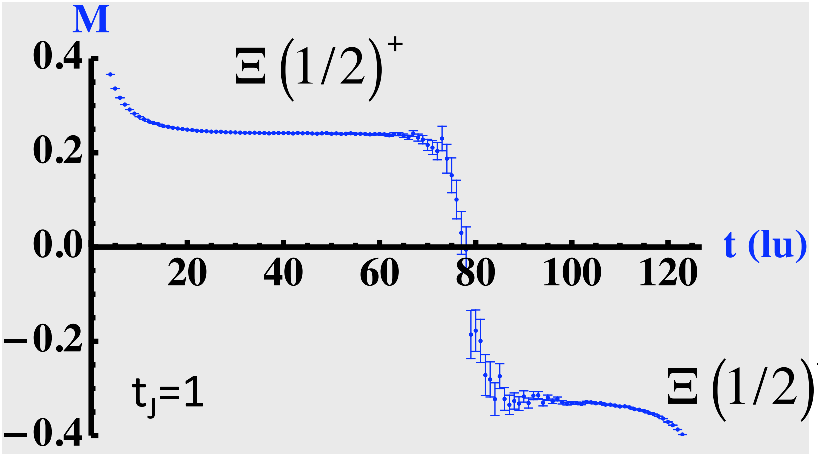


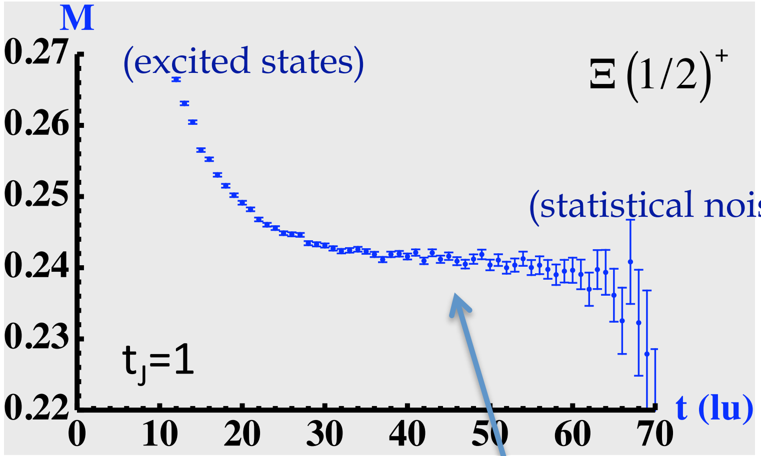
NPLQCD, Phys. Rev. D 79, 114502 (2009)

an example: Ξ mass (uss) n_f=2+1 20^3 x 128 clover fermions

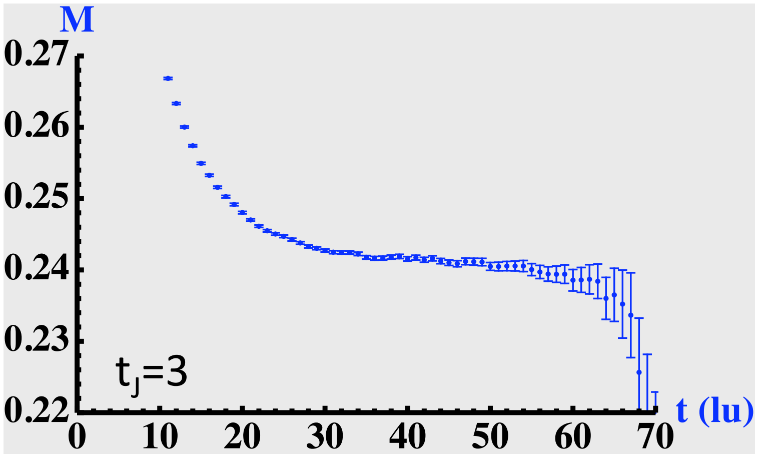
b_s=0.1227±0.0008 fm, $b_{s/}b_t$ = 3.5, L ~ 2.5 fm, m_π~390 MeV, m_K~546 MeV

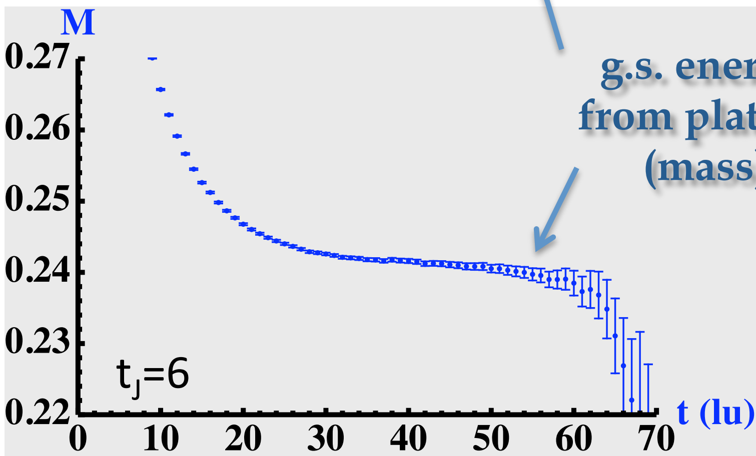
1195 configs (x 364 sources) SS (sink & source)





$$\frac{1}{t_J}\log\frac{C_A(t)}{C_A(t+t_J)} = m_A$$ Generalized effective plots

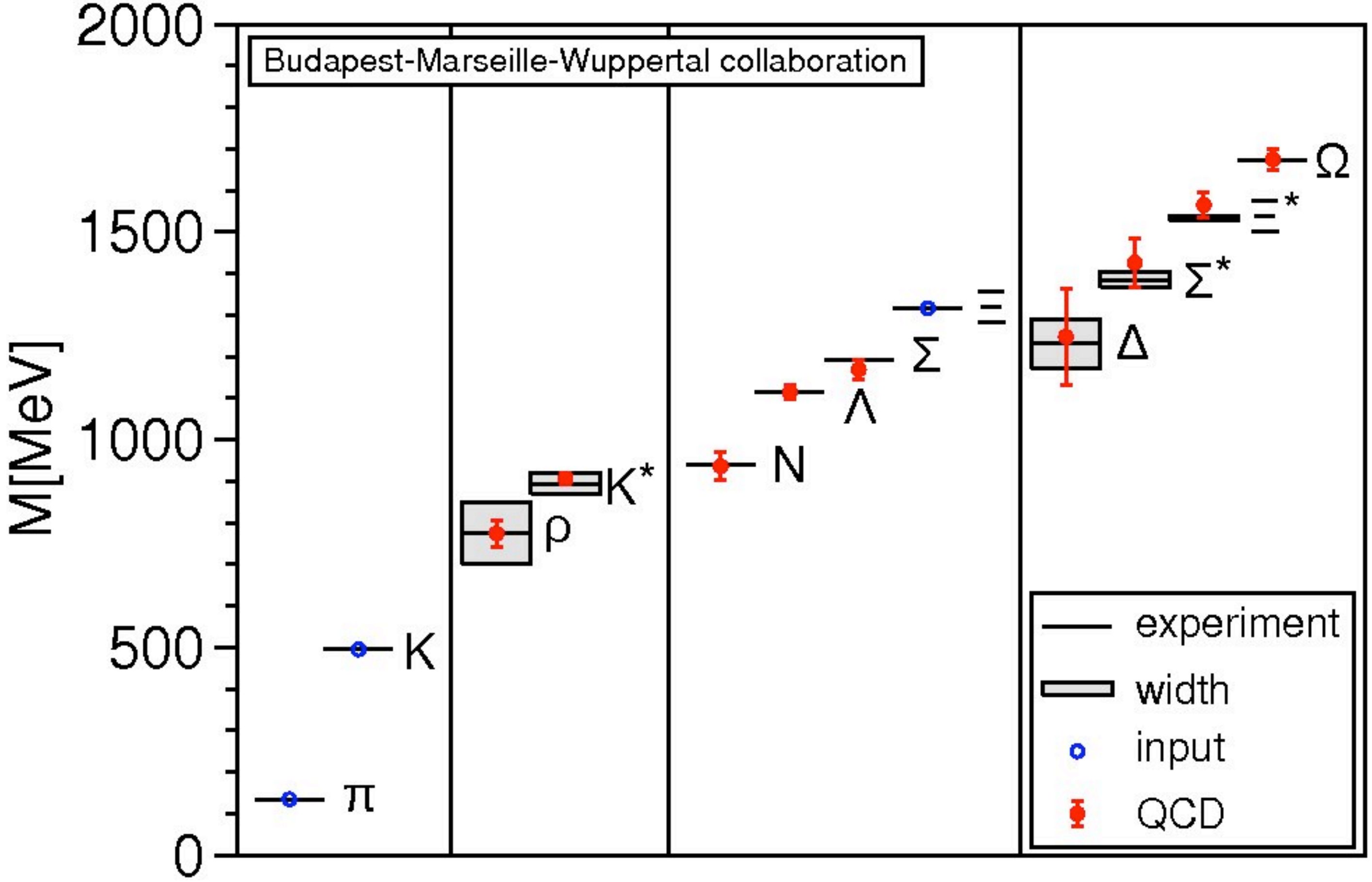




g.s. energy from plateau (mass)

(anti-periodic bc in time)

Lattice QCD works really well when calculating the static properties of hadrons, as one can see from the hadronic mass spectrum representation.



$m_\pi \sim 190$ MeV
$L \sim 4\, m_\pi^{-1} \sim 5.7$ fm

Durr et al. (BMW Collaboration) Science 322 (2008) 1224

Two-particle correlators → Energy of the interacting 2-particle system

$$C_{H_A H_B,\Gamma}(\vec{p}_1,\vec{p}_2,t) = \sum_{\vec{x}_1\vec{x}_2} e^{i\vec{p}_1\vec{x}_1} e^{i\vec{p}_2\vec{x}_2} \Gamma^{\beta_1\beta_2}_{\alpha_1\alpha_2} \left\langle J_{H_A,\alpha_1}(\vec{x}_1,t) J_{H_B,\alpha_2}(\vec{x}_2,t) \bar{J}_{H_A,\beta_1}(x_0,0) \bar{J}_{H_B,\beta_2}(x_0,0) \right\rangle$$

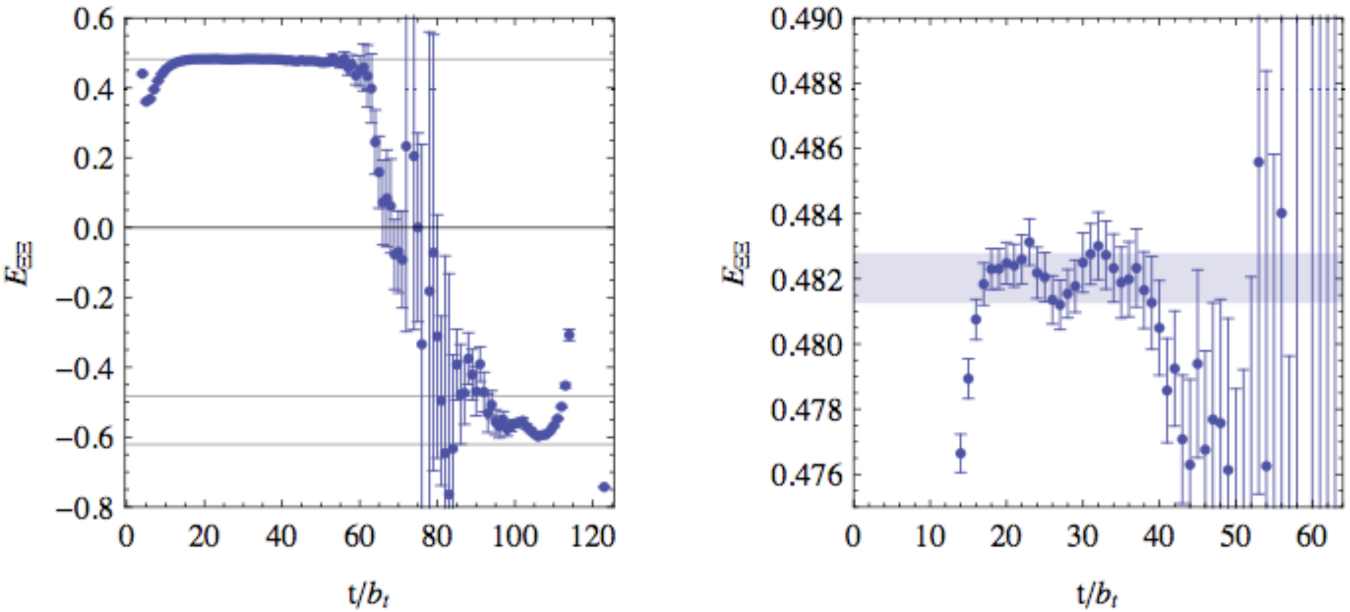
spin tensor

interpolating operator

source located at t=0

away from the source, i.e. at large t

$$C_{H_A H_B}(\vec{p},-\vec{p},t) \sim \sum_n Z^{(i)}_{n;AB}(\vec{p})\, Z^{(f)}_{n;AB}(\vec{p})\, e^{-E_n^{AB}(\vec{0})t}$$



noise-to-signal

Lepage, 1989

system of n-pions

$$\langle C(t)\rangle = \left\langle \left(\sum_x \pi^-(\vec{x},t)\right)^n \left(\pi^+(\vec{0},0)\right)^n \right\rangle \;\rightarrow\; A_0\, e^{-n\, m_\pi t}$$

$$N\sigma^2 \sim \langle C^+(t)\, C(t)\rangle - \langle C(t)\rangle^2$$

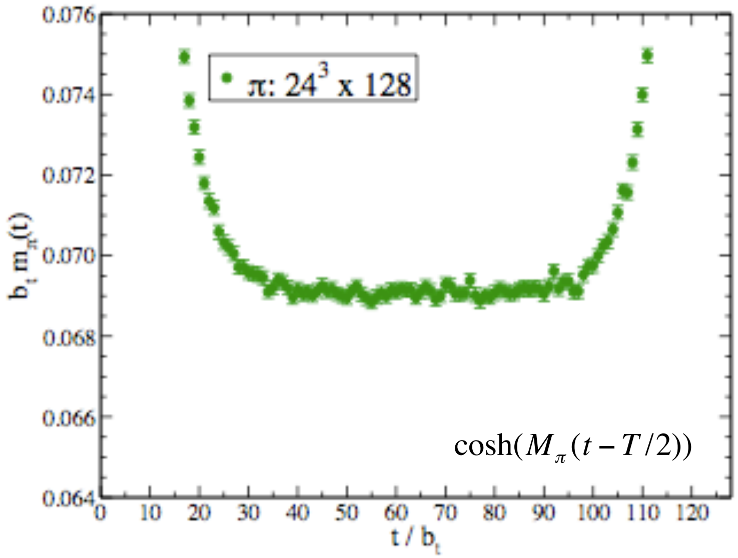
$$= \left\langle \left(\sum_x \pi^-(\vec{x},t)\right)^n \left(\sum_y \pi^+(\vec{y},t)\right)^n \left(\pi^+(\vec{0},0)\right)^n \left(\pi^-(\vec{0},0)\right)^n \right\rangle - \left\langle \left(\sum_x \pi^-(\vec{x},t)\right)^n \left(\pi^+(\vec{0},0)\right)^n \right\rangle^2$$

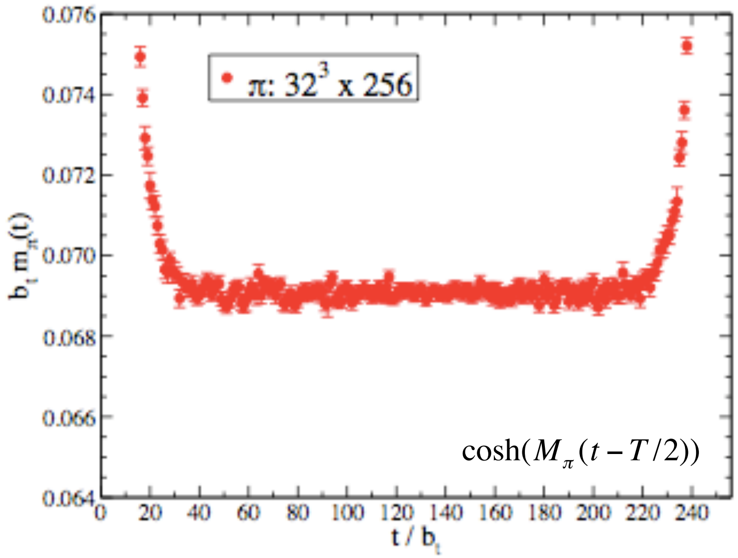
$$\rightarrow \left(A_2 - A_0^2\right) e^{-2n\, m_\pi t}$$

$$\Rightarrow\quad \boxed{\frac{\sigma(t)}{\langle C(t)\rangle} \sim \frac{\sqrt{\left(A_2 - A_0^2\right)}\; e^{-n\, m_\pi t}}{\sqrt{N}\, A_0\, e^{-n\, m_\pi t}} \sim \frac{1}{\sqrt{N}}}$$

noise-to-signal independent of time

NPLQCD, arXiv:1104.4101 [hep-lat]





noise-to-signal

Lepage, 1989

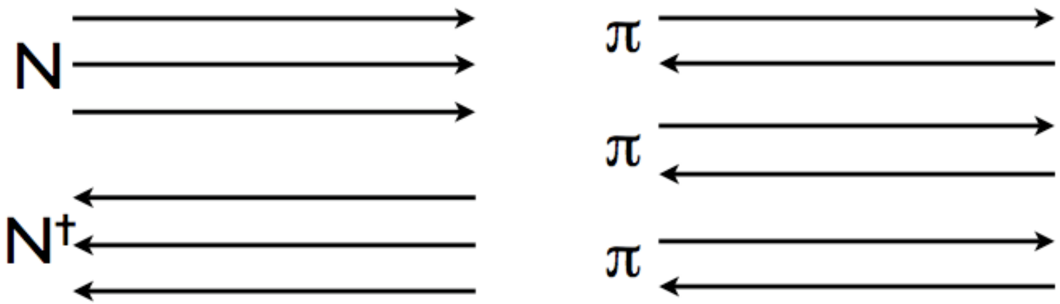
nucleons

$$\sigma^2(C) = \langle CC^\dagger \rangle - |\langle C \rangle|^2$$

$$\frac{\sigma}{\langle C \rangle} \sim \exp\left\{-\left(M_N - \frac{3m_\pi}{2}\right)t\right\}$$

$$\frac{\sigma}{\langle C \rangle} \sim \exp\left\{-A\left(M_N - \frac{3m_\pi}{2}\right)t\right\}$$

nucleus with A nucleons



exponential grow of noise

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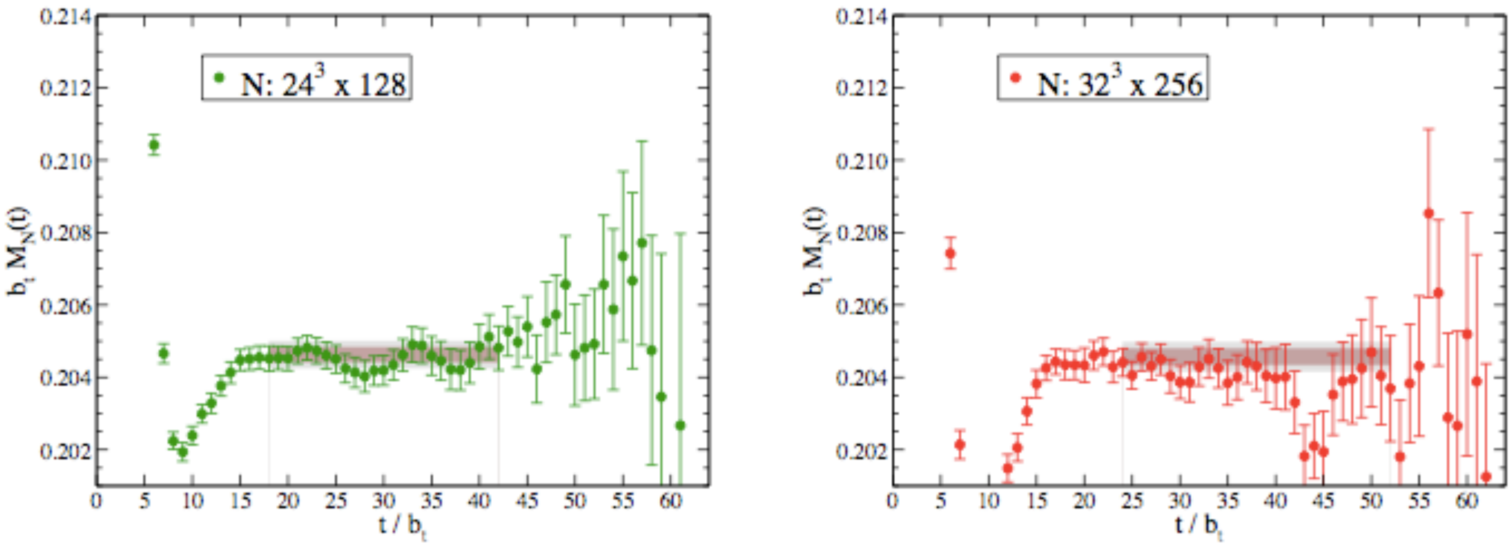
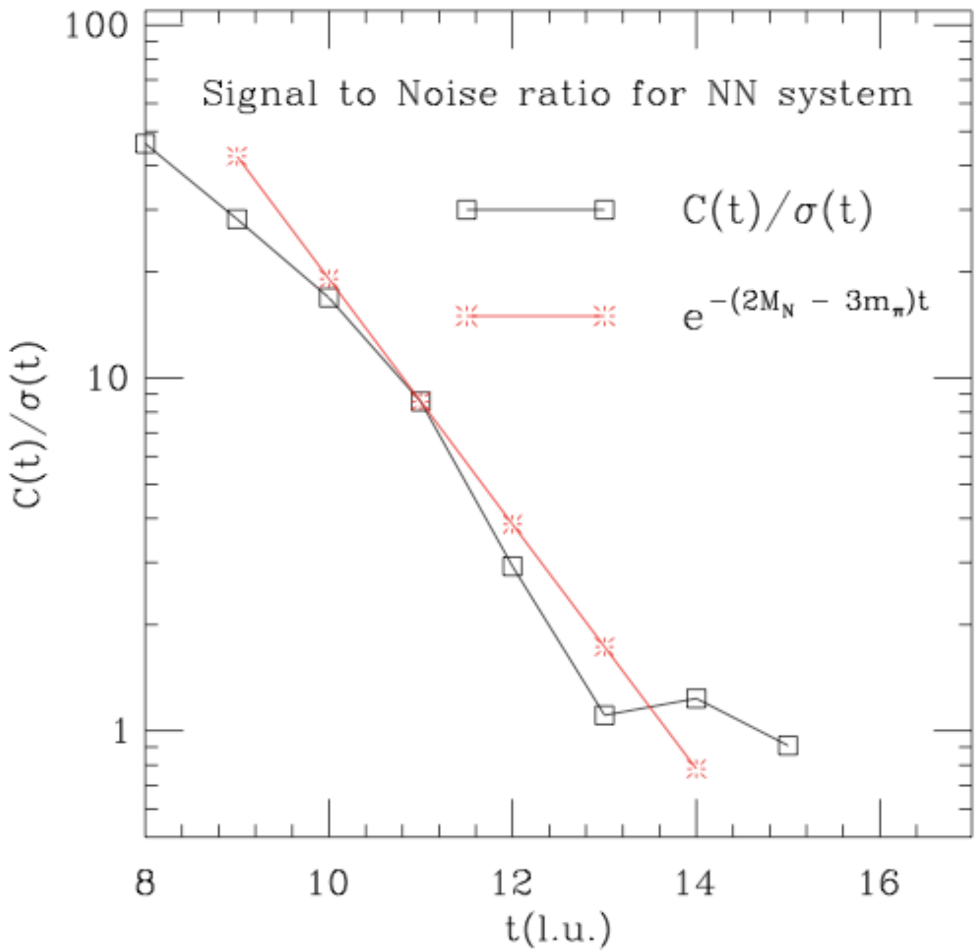
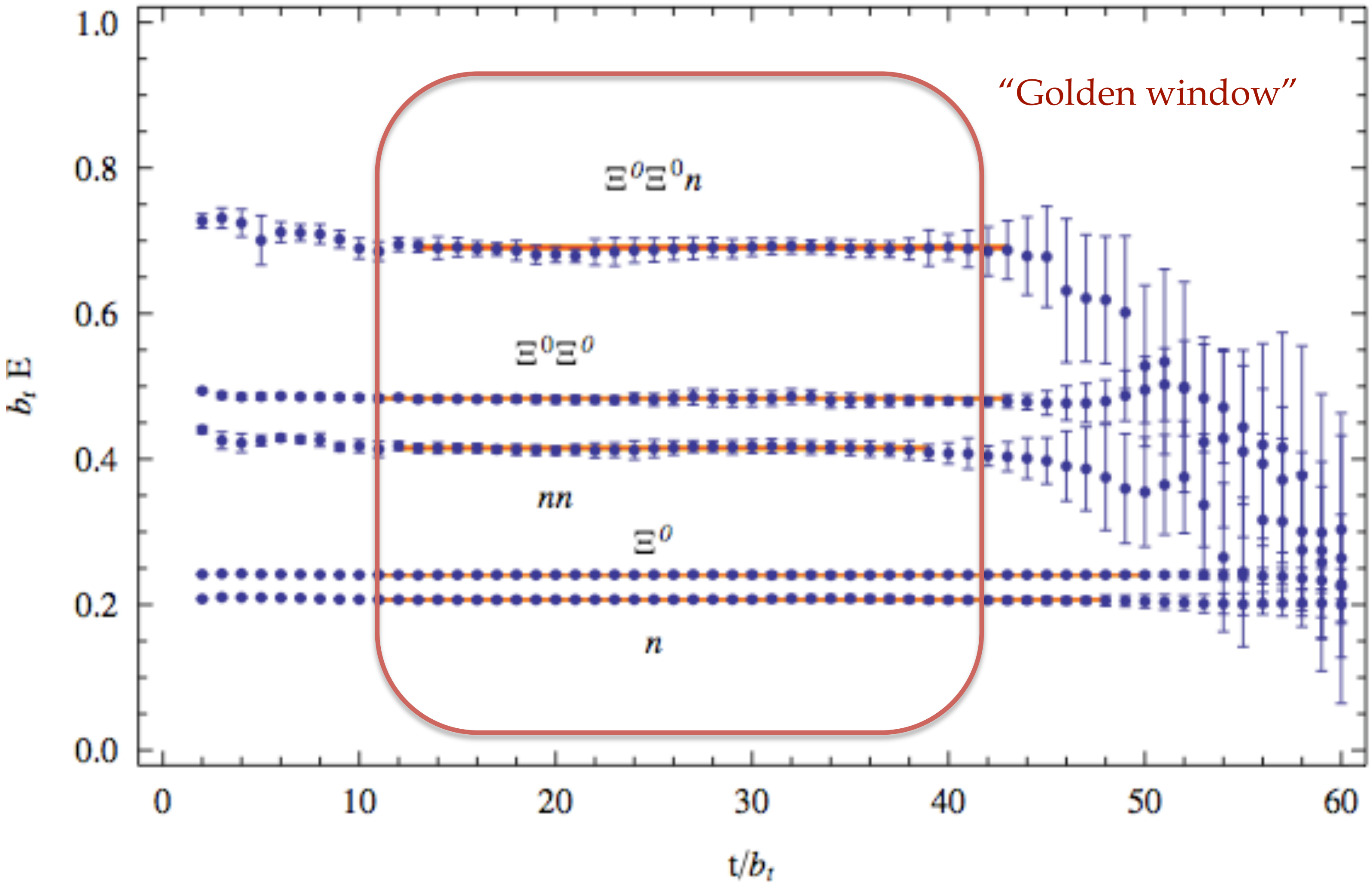
NPLQCD, arXiv:1104.4101 [hep-lat]



$np\ (^1S_0)$ NPLQCD MILC/2064f21b676m010m050



(Courtesy of P. Bedaque and A. Walker-Loud)



"Golden window"
$\Xi^0\Xi^0 n$
$\Xi^0\Xi^0$
nn
Ξ^0
n
b_t E
t/b_t
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50
60

But... how can we get the effective parameters for the low energy interaction?

The Maiani-Testa theorem states that one cannot extract multi-hadron S-matrix elements from Euclidean space Green functions at infinite volume except for kinematical thresholds.

Lüscher (1986) circumvented this problem by going to finite volume: extract the scattering length from the volume dependence of two-particle energy levels at finite volume (up to inelastic thresholds)

One-hadron correlator: $C_A(t) = \sum_{\vec{x}} \left\langle A(t,\vec{x})\, A^{\dagger}(0,\vec{0}) \right\rangle = \sum_n C_A^n\, e^{-E_A^n t} \longrightarrow C_A\, e^{-M_A t}$

energies

Two-hadron correlator:

$$C_{AB}(t) = \sum_{\vec{x},\vec{y}} \left\langle A(t,\vec{x})\, B(t,\vec{x})\, B^{\dagger}(0,\vec{0})\, A^{\dagger}(0,\vec{0}) \right\rangle = \sum_n C_{AB}^n\, e^{-E_{AB}^n t} \longrightarrow C_{AB}\, e^{-E_{AB} t}$$

Energy shift: $\Delta E = E_{AB} - M_A - M_B$

$$G_{AB}(t) = \frac{C_{AB}(t)}{C_A(t)\, C_B(t)} = \sum_n C^n e^{-\Delta E^n t} \longrightarrow \boxed{C e^{-\Delta E t}}$$

below inelastic thresholds

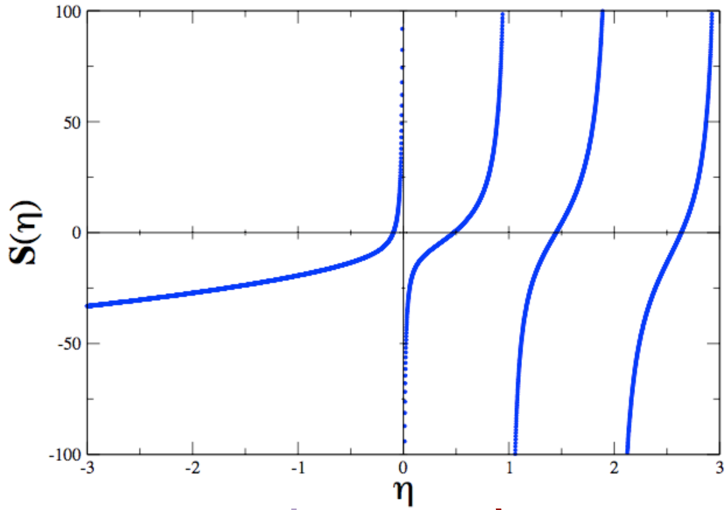
$$\Delta E \equiv \sqrt{p^2 + M_A^2} + \sqrt{p^2 + M_B^2} - M_A - M_B$$

non-interacting particles

$$\vec{p} = \frac{2\pi\vec{n}}{L}$$

obtained from the simulation

$$S\left(\eta^2 = \frac{p^2 L^2}{4\pi^2}\right) \equiv \sum_{j}^{|\vec{j}|<\Lambda} \frac{1}{|\vec{j}|^2 - \eta^2} - 4\pi\Lambda$$



u.v. regulator

$$\boxed{-\frac{1}{a} + \frac{1}{2} r_0 p^2 =}$$

efective range expansion

$$p \cot\delta(p) = -\frac{1}{\pi L} S\left(\frac{p^2 L^2}{4\pi^2}\right)$$

Beane, Bedaque, Parreño, Savage, Phys. Lett. B 585,1-2, 106-114 **(2004)**

$$p\cot\delta = -\frac{1}{a} + \frac{1}{2} r p^2 + \ldots$$

$$\boxed{p\cot\delta(p) = \frac{1}{\pi L} S(\eta) = \frac{1}{\pi L} \sum_{\vec{j}}^{|\vec{j}|<\Lambda} \frac{1}{|\vec{j}|^2 - \eta^2} - \frac{4\Lambda}{L}}$$

with $\eta = \dfrac{L}{2\pi} p$

If one retains only the scattering length in the tan(δ) expansion and performing a perturbative expansion on the momentum p^2 (and playing a bit with the sums...)

$$\Delta E = \frac{p^2}{M} = \frac{4\pi a}{M L^3}\left[1 - c_1 \frac{a}{L} + c_2 \left(\frac{a}{L}\right)^2 + \ldots\right]$$

with $c_1 = \dfrac{1}{\pi} \sum_{\substack{\vec{j} \in Z^3 \\ \vec{j} \neq \vec{0}}} \dfrac{1}{|\vec{j}|^2}$, $\quad c_2 = c_1^2 - \dfrac{1}{\pi^2} \sum_{\vec{j} \neq 0} \dfrac{1}{|\vec{j}|^4}$

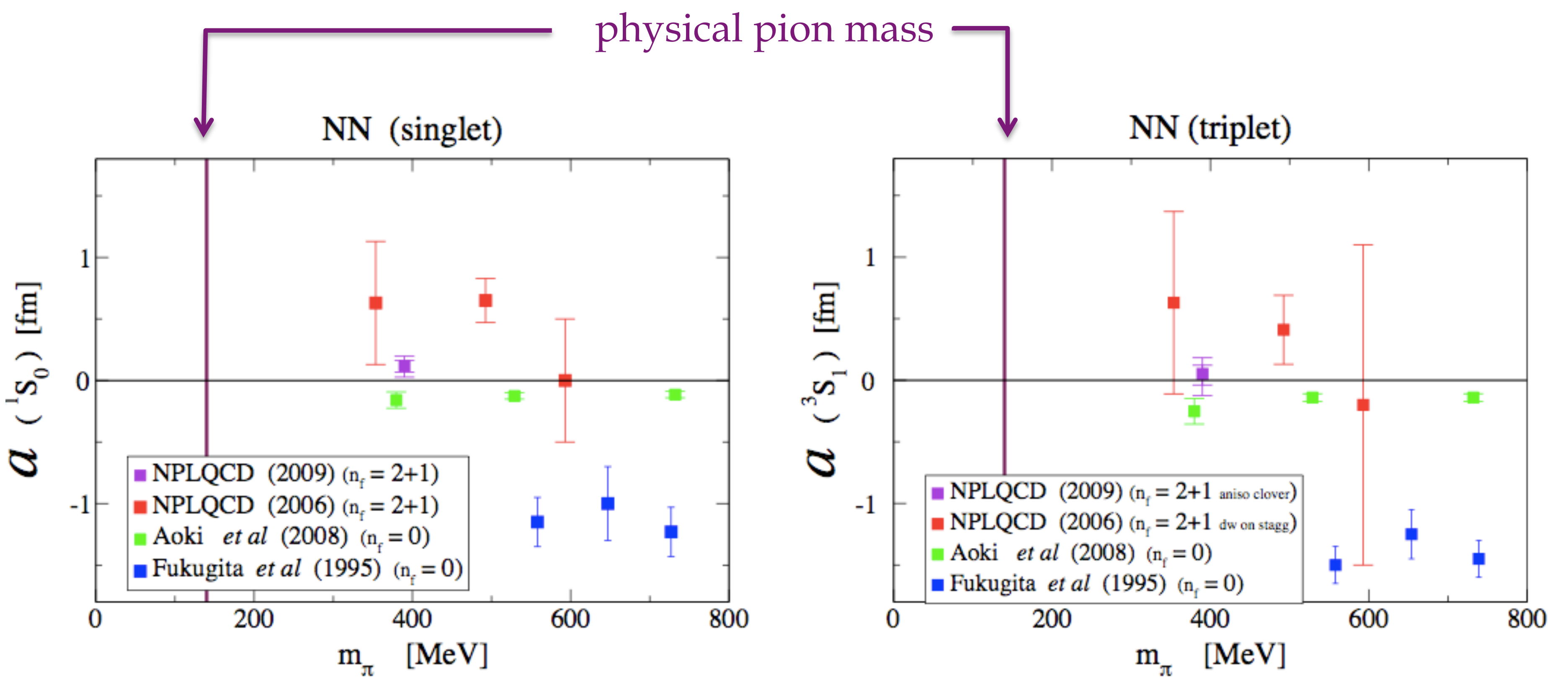
(recovering Lüscher's formula, M. Lüscher, Commun. Math. Phys. 105, 153 (1986))

For negatively energy shifted states (in the lattice volume):

$$p = i\,\kappa, \qquad \kappa = \gamma + \frac{g_1}{L}\left(e^{-\gamma L} + \sqrt{2} e^{-\sqrt{2}\gamma L} + \ldots\right) \qquad \text{with } \gamma << m_\pi \qquad B.E._\infty = \frac{\gamma^2}{M}$$

(finite volume dependence suppressed exponentially by the binding momentum)

With simulations performed at two or more lattice volumes (with $p^2 < 0$) it is possible to perform an extrapolation to determine the b.e. at infinite volume, and at this pion mass.



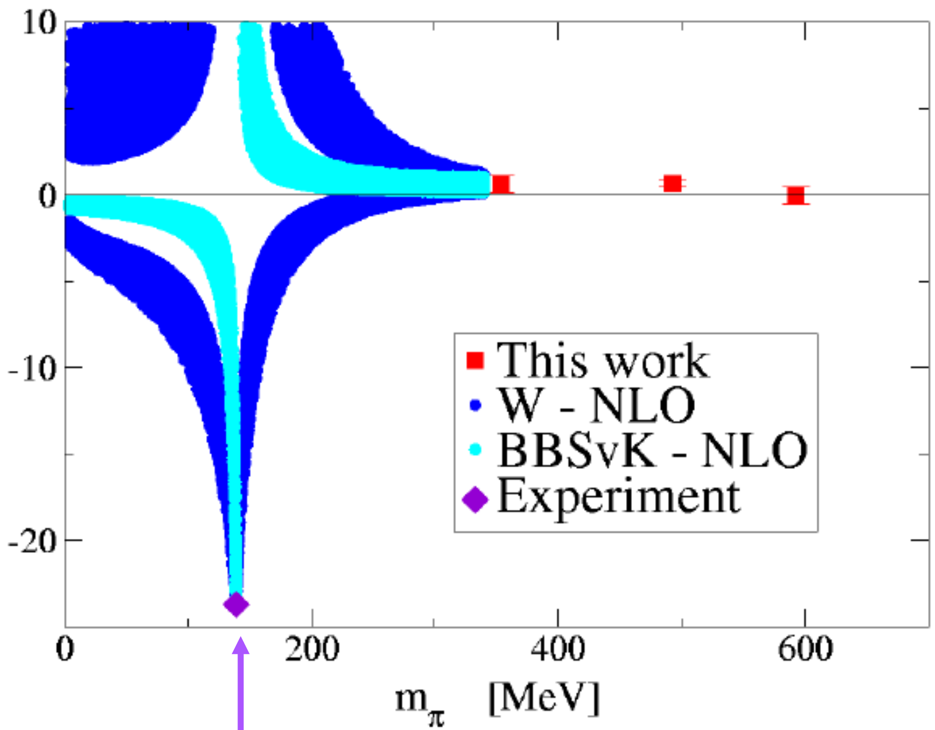
exp: - 23.8 fm

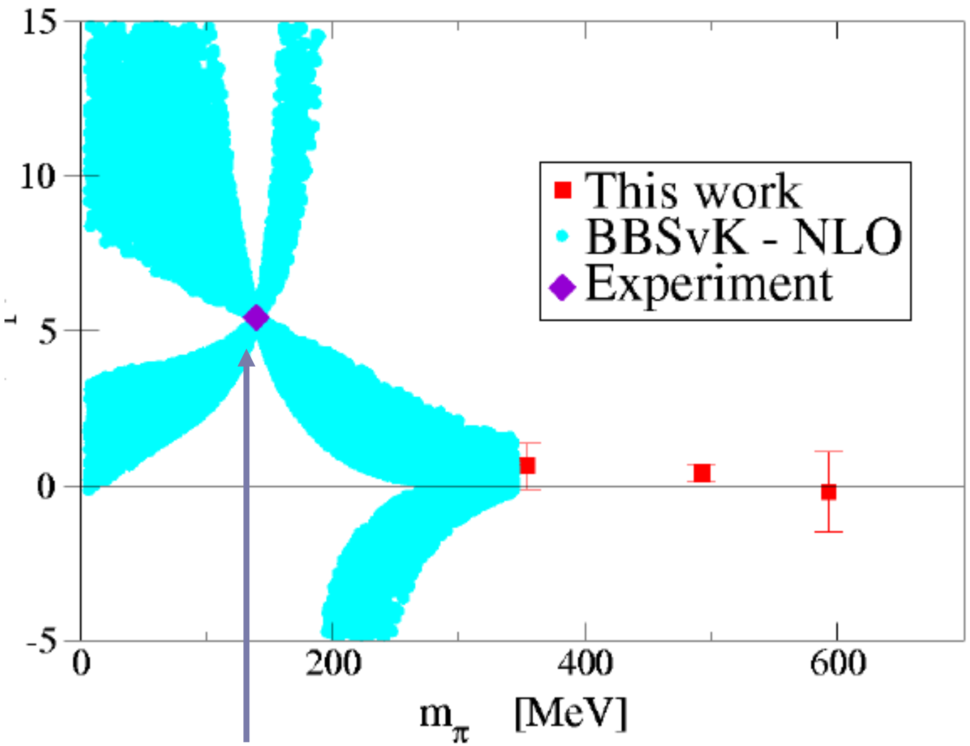
exp: 5.4 fm

compilation of the NN scattering lengths calculations for the singlet and triplet channels

NN

Beane, Bedaque, Orginos, Savage, PRL97 012001 (2006)





a ($^1S_0$) [fm]

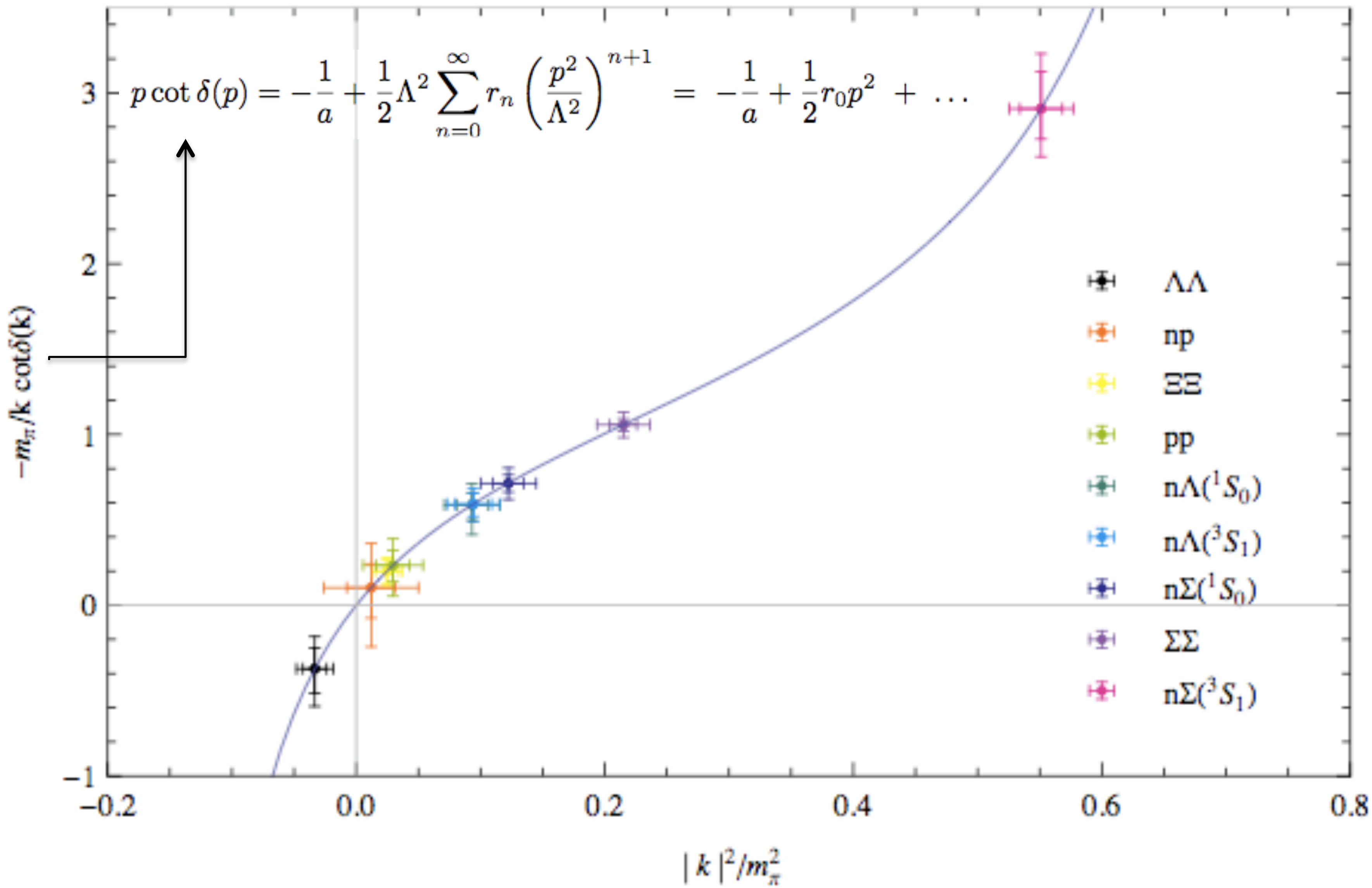
5.4 fm

a ($^3S_1$) [fm]

- 23.8 fm

@ m_π = 350, 590, 590 MeV

L=2.5 fm

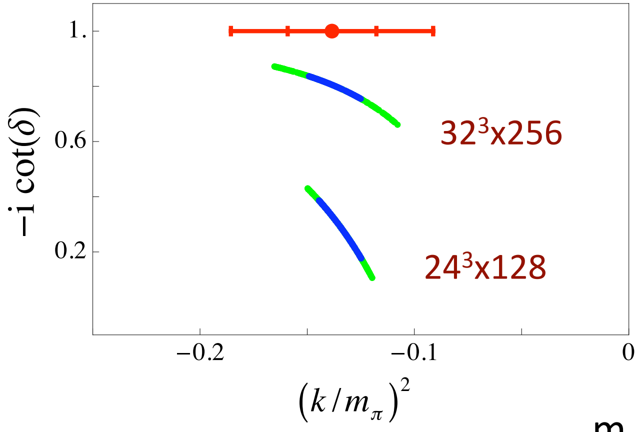


NPLQCD Collaboration

Bound states?

H-dibaryon $(\Lambda\Lambda)_{J=0,\, I=0}$

Phys. Rev. Lett. 106 (2011) 162001

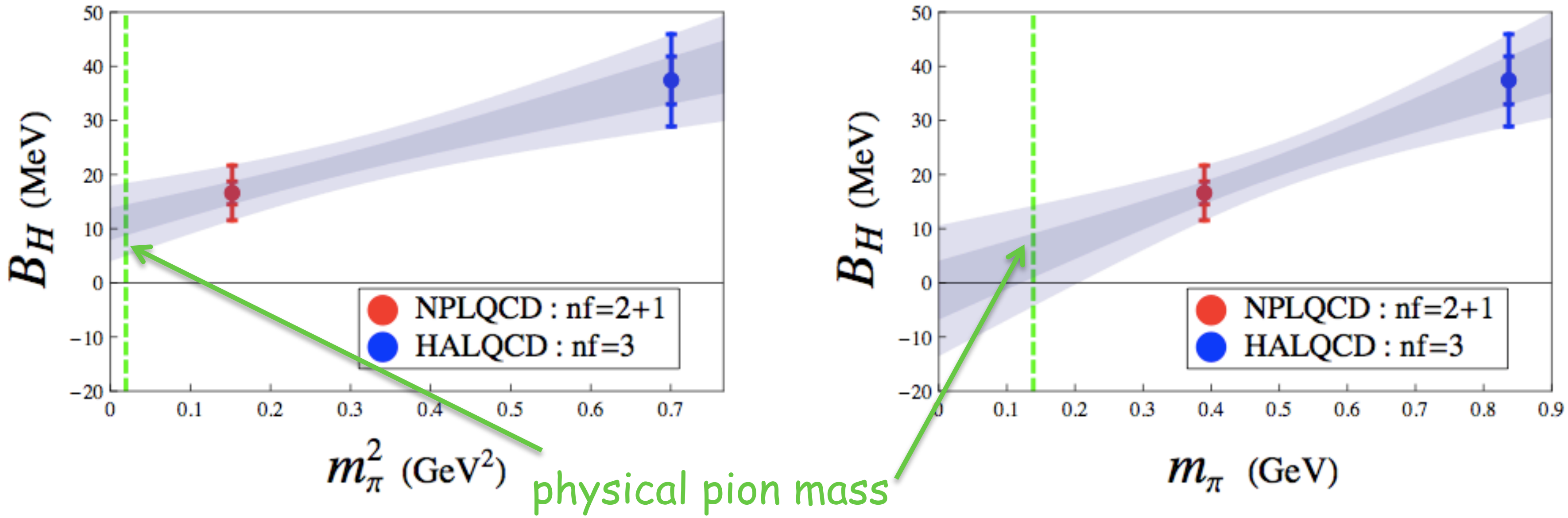


m_π~390 MeV

$$f(\Omega) = \frac{1}{k\cot\delta(k) - ik} \quad \to \infty \quad (\text{b.s.}) \qquad -i\ \cot\delta(k) = 1$$

$$\kappa = \gamma + \frac{g_1}{L}\left(e^{-\gamma L} + \sqrt{2}e^{-\sqrt{2}\gamma L} + \ldots\right) \qquad B_\infty^H = \frac{\gamma^2}{M}$$

$$\boxed{B_\infty^H = 16.6 \pm 2.1 \pm 4.5 \pm 1.0 \pm 0.6\ \text{MeV}}$$



... more simulations at lighter quark masses (m_π~200–250 MeV)

But Nuclear Physics mean more than 2 particles...
Can we handdle this with Lattice QCD?

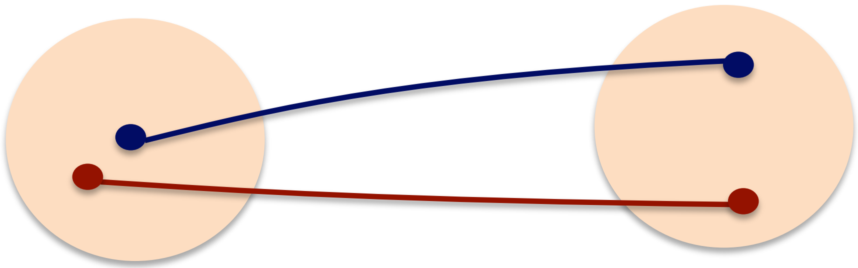
But we want to do nuclear physics, i.e., we need to simulate systems with larger number of hadrons

scattering length

$$\Delta E_n = \frac{4\pi \boxed{a}}{ML^3}\binom{n}{2}\left\{1 - \frac{\boxed{a}\mathcal{I}}{\pi L} + \left(\frac{\boxed{a}}{\pi L}\right)^2\left[\mathcal{I}^2 + (2n-5)\mathcal{J}\right] - \left(\frac{\boxed{a}}{\pi L}\right)^3\left[\mathcal{I}^3 + (2n-7)\mathcal{I}\mathcal{J} + (5n^2 - 41n + 63)\mathcal{K}\right]\right\}$$
$$+ \binom{n}{2}\frac{8\pi^2 \boxed{a^3}}{ML^6} r + \binom{n}{3}\frac{\bar{\eta}_3^L}{L^6} + \mathcal{O}(1/L^7),$$

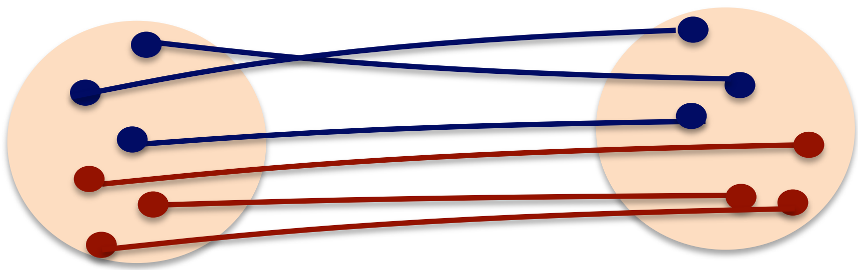
contains the 3-particle interaction

[Bogoliubov '47][Huang,Yang '57][Beane, Detmold, Savage PRD76;074507, 2007; Detmold,Savage PRD77:057502,2008]



$$C_{n\pi^+}(t) = \left\langle \sum_x \pi^+(\vec{x},t)\, \pi^-(\vec{0},0) \right\rangle \xrightarrow{\text{large t}} A e^{-E\,t}$$

$$\pi^+(\vec{x},t) = \bar{d}(\vec{x},t)\gamma_5 u(\vec{x},t)$$



$$C_{n\pi^+}(t) = \left\langle \left(\sum_x \pi^+(\vec{x},t)\right)^n \left(\pi^-(\vec{0},0)\right)^n \right\rangle \xrightarrow{\text{large t}} A e^{-E_n t}$$

But we want to do nuclear physics, i.e., we need to simulate systems with larger number of hadrons

scattering length

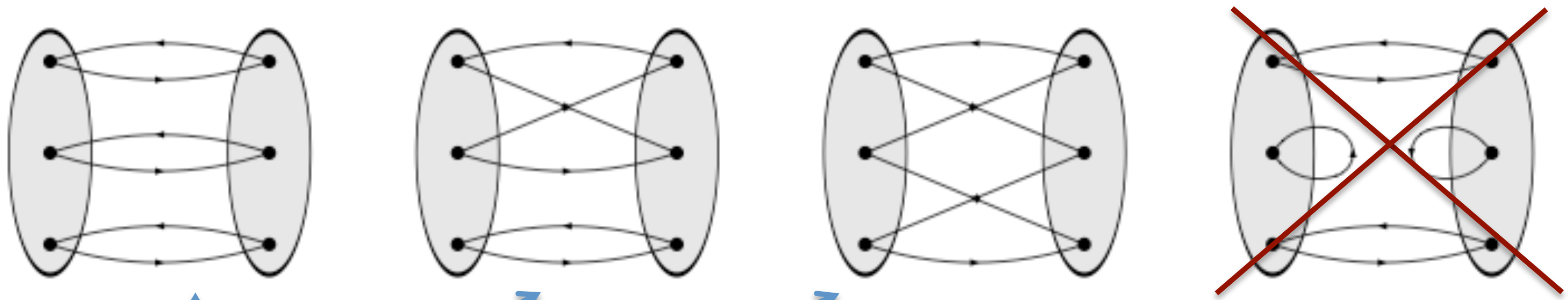
$$\Delta E_n = \frac{4\pi \boxed{a}}{ML^3}\binom{n}{2}\left\{1 - \frac{\boxed{a}\mathcal{I}}{\pi L} + \left(\frac{\boxed{a}}{\pi L}\right)^2\left[\mathcal{I}^2 + (2n-5)\mathcal{J}\right] - \left(\frac{\boxed{a}}{\pi L}\right)^3\left[\mathcal{I}^3 + (2n-7)\mathcal{I}\mathcal{J} + (5n^2 - 41n + 63)\mathcal{K}\right]\right\}$$
$$+ \binom{n}{2}\frac{8\pi^2 \boxed{a^3}}{ML^6} r + \binom{n}{3}\frac{\bar{\eta}_3^L}{L^6} + \mathcal{O}(1/L^7),$$

contains the 3-particle interaction

[Bogoliubov '47][Huang,Yang '57][Beane, Detmold, Savage PRD76;074507, 2007; Detmold,Savage PRD77:057502,2008]

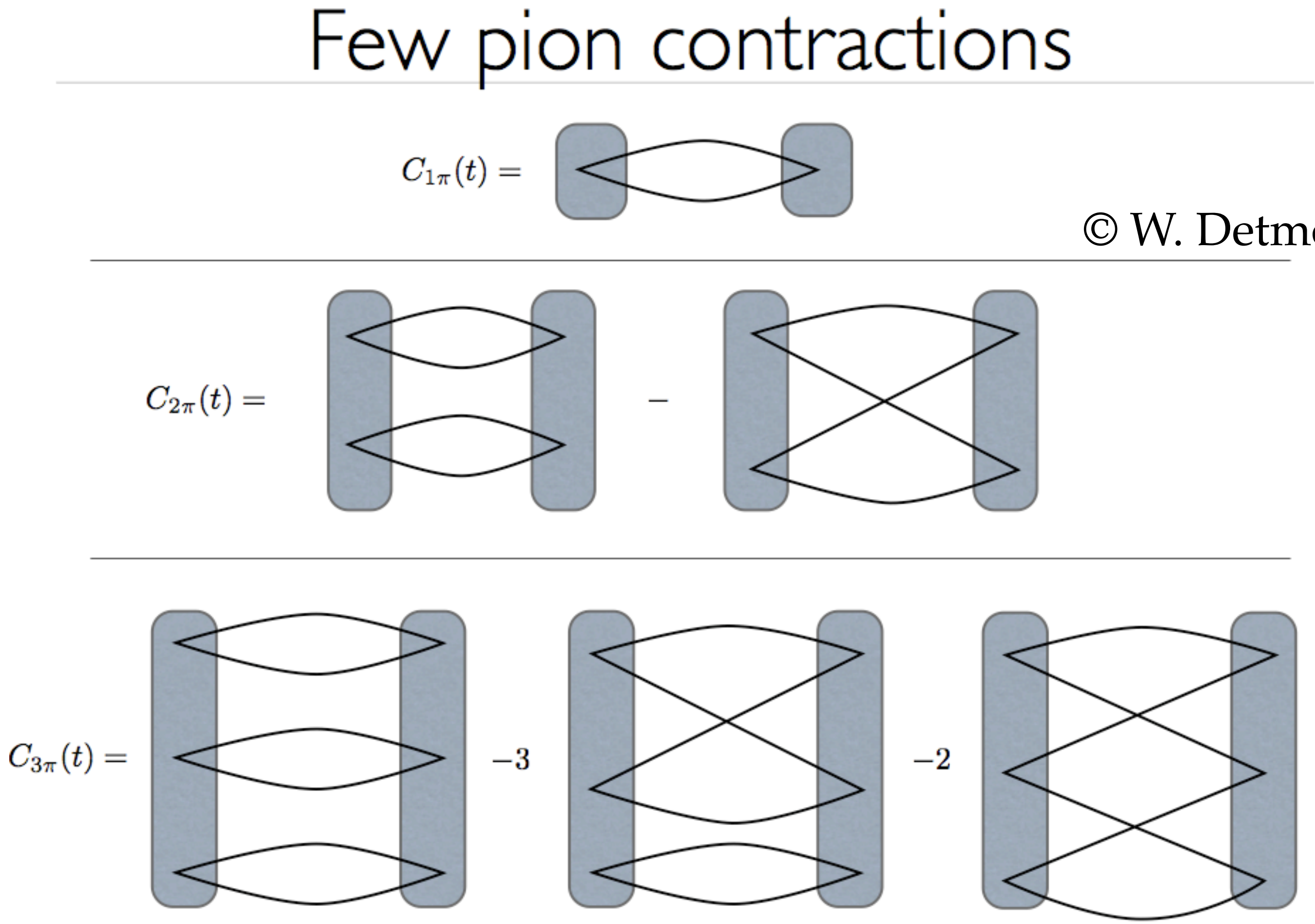
$$C_{n\pi^+}(t) = \left\langle \left(\sum_x \pi^+(\vec{x},t)\right)^n \left(\pi^-(\vec{0},0)\right)^n \right\rangle \xrightarrow{\text{large t}} A e^{-E_n t} \qquad \text{where } \pi^+(\vec{x},t) = \bar{d}(\vec{x},t)\gamma_5 u(\vec{x},t)$$

Ex. 3 pions (maximal isospin, I_z=3)



$$C_3(t) = \mathrm{Tr}[\Pi]^3 - 3\,\mathrm{Tr}[\Pi]\mathrm{Tr}[\Pi^2] + 2\mathrm{Tr}[\Pi^3] \quad \text{with} \quad \Pi = \sum_x \gamma_5 S(\vec{x},t;\vec{0},0)\gamma_5 S^+(\vec{x},t;\vec{0},0)$$

increasing complexity of performing contractions with the number of hadrons
This is the real bootleneck for doing nuclear physics



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Exercise: for a given number of hadrons, composed by u,d, and s quarks, what would be (naively) the number of contractions one would have to perform?

Something for you to think about... and discuss in the afternoon session

Make a list with those challenges you think Lattice QCD faces in order to impact the field of Nuclear Physics?

This was a very general presentation of the application of Lattice technology to nuclear physics phenomena.

Many aspects were not covered:

The interpolating fields create point sources, with little overlap with hadrons, which have a size (~ 1 fm). From the simulation point of view, how do we optimize the projection onto the state we are interested in?

Which are the analysis techniques that are being used to isolate the ground state from our simulation *data*?

Is there a way to deal with the large number of contractions which appear in LQCD simulations of many-hadron systems?

Please, contact me if you want to know more about these (and other related) points.

Acknowledgents:

Most of the information shown about the Lattice QCD formalism comes from the following texts and presentations you can find easily on the web:

"Introduction to Lattice QCD", Rajan Gupta
"Introduction to lattice QCD", Constantia Alexandrou
"Introduction to lattice QCD", Marco Panero
"From Monte Carlo Integration to Lattice Quantum Chromo Dynamics", Massimo Di Pierro
"Computation of meson masses on the lattice", Roman Wielsing