Unitary fermions, conformal symmetry and holography

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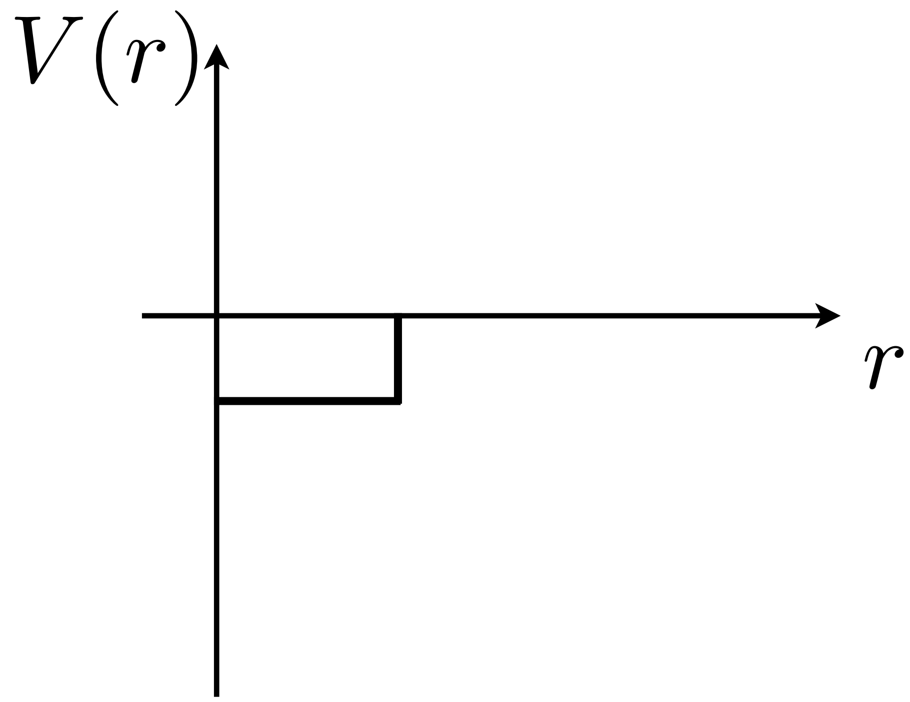
Plan of the talk

- Fermions at unitarity
- Conformal invariance
- Classification of operators: primary, descendants
- Operator-state correspondence
- OPEs
- Holography

Ref: Nishida and DTS, arXiv: 0706.3746

Fermions at unitarity

Consider 2 particles interacting though a potentials



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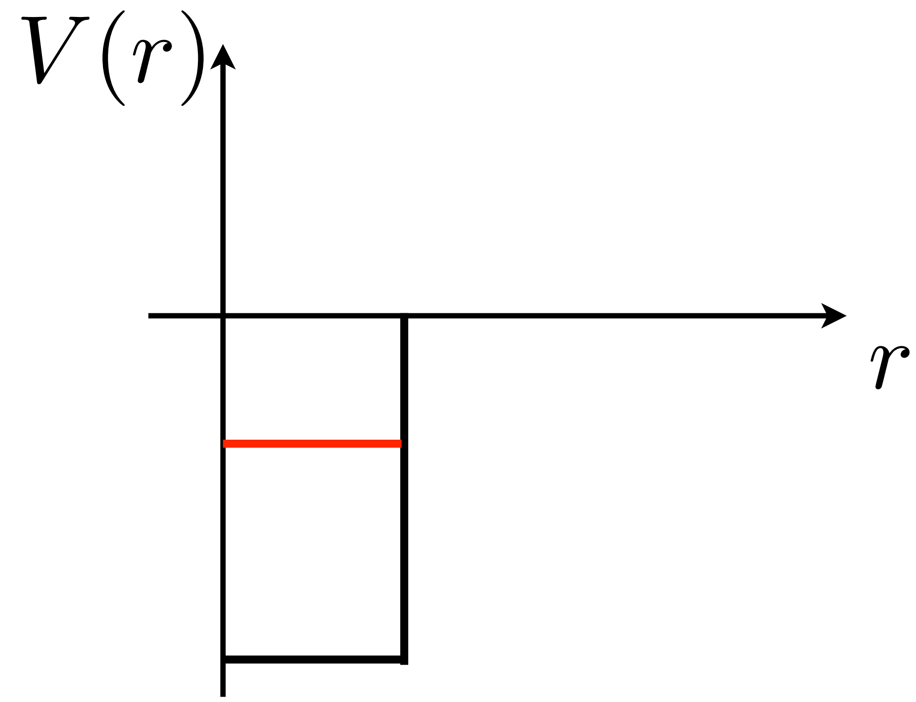

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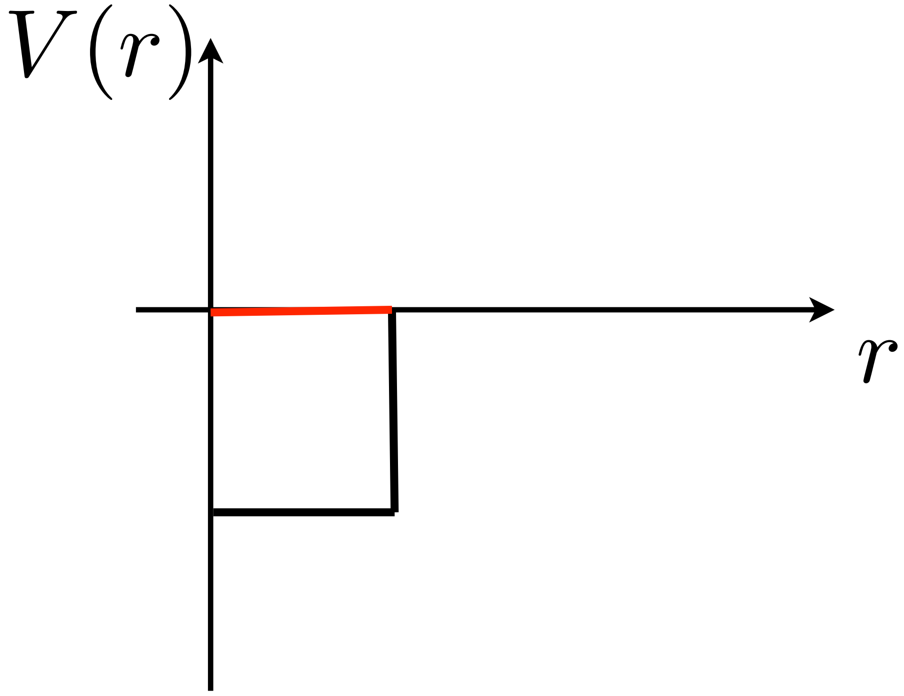
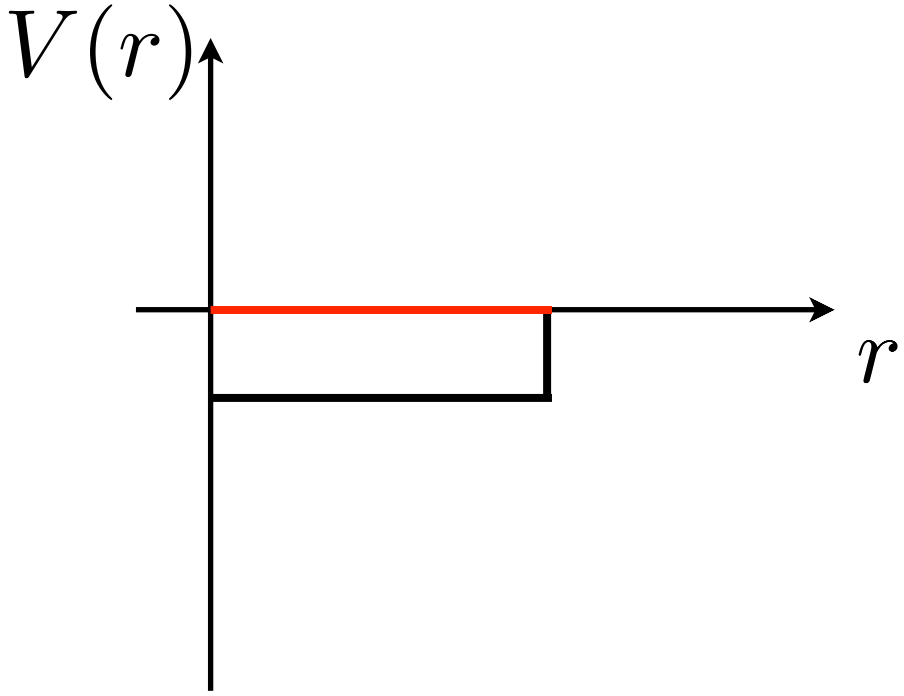


Zero-range limit



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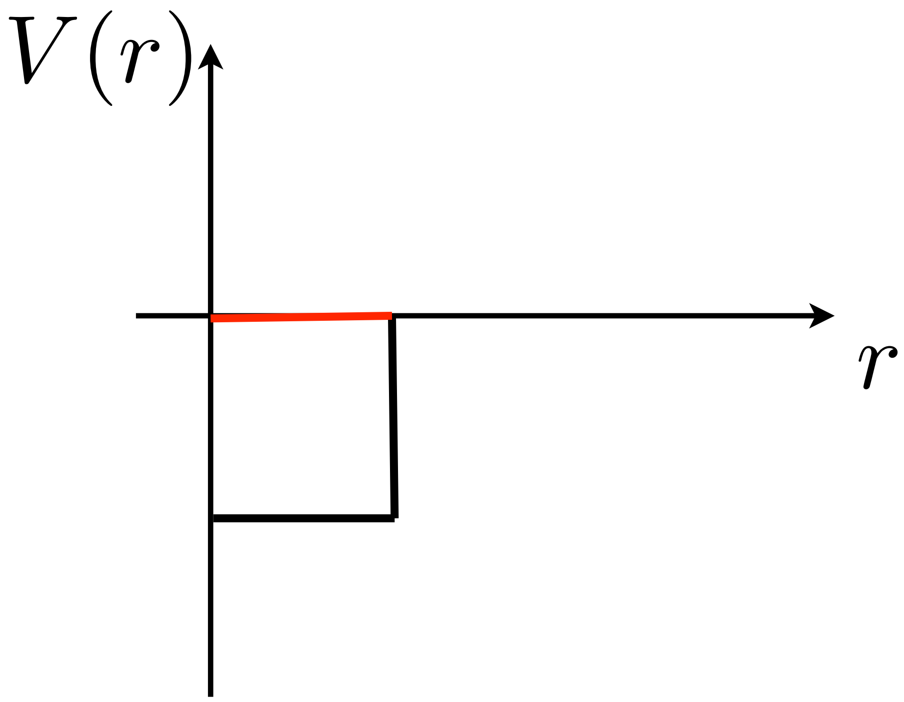

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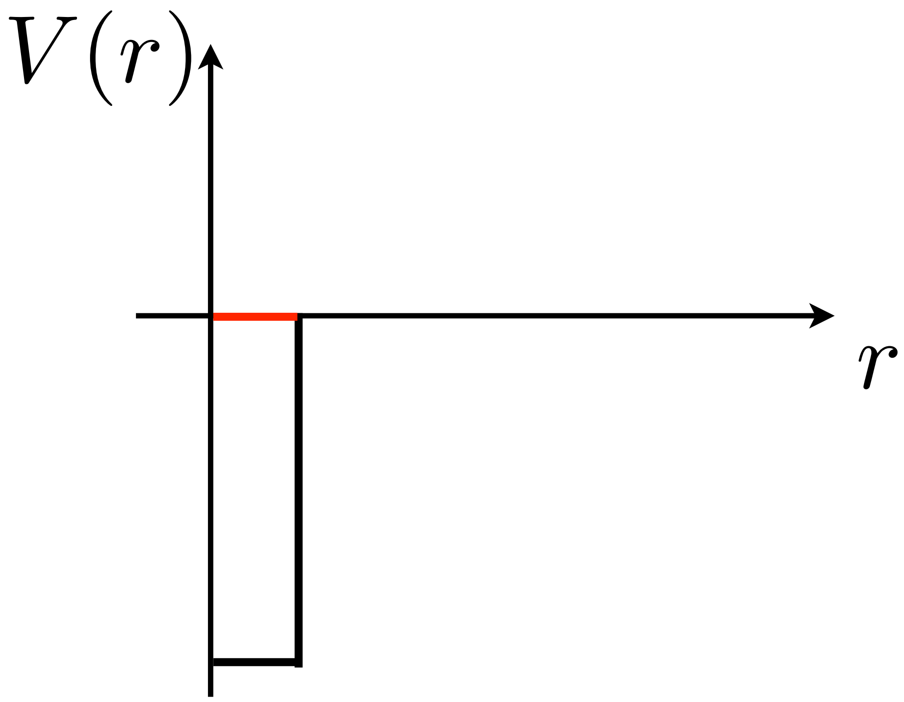


Physical realizations

- Neutrons: $a = -20$ fm $>> 1/m_\pi$
- Trapped atomic systems, fine-tuning done by external magnetic field

Boundary conditions

Unitarity fermions = system described by free Hamiltonian

$$H = \sum_i \frac{\mathbf{p}_i^2}{2m}$$

with nontrivial boundary condition on the wave function:

$$\Psi(\underbrace{\mathbf{x}_1, \mathbf{x}_2, \ldots}_{\text{spin}-\text{up}}, \underbrace{\mathbf{y}_1, \mathbf{y}_2, \ldots}_{\text{spin}-\text{down}})$$

$$\Psi \to \frac{C}{|\mathbf{x}_i - \mathbf{y}_j|} + 0 \times |\mathbf{x}_i - \mathbf{y}_j|^0 + O(|\mathbf{x}_i - \mathbf{y}_j|) \qquad |\mathbf{x}_i - \mathbf{y}_j| \to 0$$

Free fermions corresponds to another boundary condition:

$$\Psi \to \frac{0}{|\mathbf{x}_i - \mathbf{y}_j|} + C + O(|\mathbf{x}_i - y_j|)$$

Bertsch parameter

- Ground state energy density ε of a unitary Fermi gas at fixed density n

$$\epsilon_{\rm unitary}(n) = \xi \epsilon_{\rm free}(n)$$

ξ is a universal parameter: Bertsch parameter

nonperturbative

Most recent Monte-Carlo evaluations: ξ≈0.4

Galilean algebra

Conserved quantities:

$$\left.\begin{array}{rll} \text{Hamiltonian:} & H & t \to t + \delta t \\ \text{Momentum:} & P_i & \mathbf{x} \to \mathbf{x} + \mathbf{a} \\ \text{Particle number (mass):} & M & \psi \to e^{i\alpha}\psi \end{array}\right\} \text{commute}$$

$N = M$

and Galilean boosts: K_i $\mathbf{x} \to \mathbf{x} + \mathbf{v}t$

M, P, K can be expressed in terms of local density and current:

$$M = \int d\mathbf{x}\, n(\mathbf{x}) \qquad \mathbf{P} = \int d\mathbf{x}\, \mathbf{j}(\mathbf{x}) \qquad \mathbf{K} = \int d\mathbf{x}\, \mathbf{x}\, n(\mathbf{x})$$

$$n = \psi^\dagger \psi, \quad \mathbf{j} = -\frac{i}{2}(\psi^\dagger \nabla \psi - \nabla \psi^\dagger \psi)$$

also angular momentum

Galilean algebra (II)

Using commutation relations between n and j:

$$[n(\boldsymbol{x}),\, n(\boldsymbol{y})] = 0, \qquad [n(\boldsymbol{x}),\, j_i(\boldsymbol{y})] = -in(\boldsymbol{y})\partial_i\delta(\boldsymbol{x}-\boldsymbol{y}),$$
$$[j_i(\boldsymbol{x}),\, j_j(\boldsymbol{y})] = -i\left(j_j(\boldsymbol{x})\partial_i + j_i(\boldsymbol{y})\partial_j\right)\delta(\boldsymbol{x}-\boldsymbol{y}).$$

Landau 1941

and $\quad [H,\, n] = -i\partial_t n = i\nabla\cdot\mathbf{j}$

$$[K_i,\, P_j] = i\delta_{ij}M$$
$$[K_i,\, H] = iP_i$$

Homework: prove these commutators

Other commutators are zero

Note: K is not conserved, but Galilean invariance has physical consequences: generating family of solutions

Scale invariance

$$\mathbf{x} \to \lambda \mathbf{x}, \quad t \to \lambda^2 t$$

Should be an invariance of fermions at unitarity: no length scale

$$D = \int d\mathbf{x}\, \mathbf{x} \cdot \mathbf{j}$$

$$[D,\, O] = i\Delta_O O$$

dim of O

$$[D,\, \mathbf{P}] = i\mathbf{P}$$

$$[D,\, \mathbf{K}] = -i\mathbf{K}$$

$$[D,\, H] = 2iH$$

for scale-invariant Hamiltonian

Conformal invariance

If ψ satisfies the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(t,\mathbf{x}_i) = -\sum_i \frac{\nabla_i^2}{2m}\psi(t,\mathbf{x}_i)$$

then

$$\psi_\lambda(t,\mathbf{x}_i) = \frac{1}{(1-\lambda t)^{d/2}} \exp\left[-\frac{im\lambda}{2(1-\lambda t)}\sum_i \mathbf{x}_i^2\right]\psi\left(\frac{t}{1-\lambda t}, \frac{\mathbf{x}}{1-\lambda t}\right)$$

is also a solution to the time-dependent Schr. eq. for any λ

Short-distance boundary condition is preserved.

This property is preserved with $|x_i - x_j|^{-2}$ potential

Conformal algebra

Contain Galilean operators, dilatation D, and

$$C = \frac{1}{2} \int d\mathbf{x}\, \mathbf{x}^2 n(\mathbf{x})$$

Nonzero commutators involving C:

$$[C,\, P_i] = iK_i \qquad [D,\, C] = -2iC \qquad [C,\, H] = iD$$

$$[D,\, H] = 2iH$$

SO(2,1) subalgebra

Particle number N: center of the algebra

Homework: compute the commutators between D, C, and H

Local operators

Include ψ, $\psi^\dagger$, $\partial_i\psi$, composites like $\psi_\uparrow(x)\psi_\downarrow(x)$ which in general needs renormalization

Commutators with H and P: $$[H,\, O(t,\mathbf{x})] = -i\partial_t O(t,\mathbf{x})$$
$$[P_i,\, O(t,\mathbf{x})] = -i\partial_i O(t,\mathbf{x})$$

Classification:

Particle number: $[N,\, O(x)] = iN_O O(x)$

Dimension: $[D,\, O(0)] = i\Delta_O O(0)$

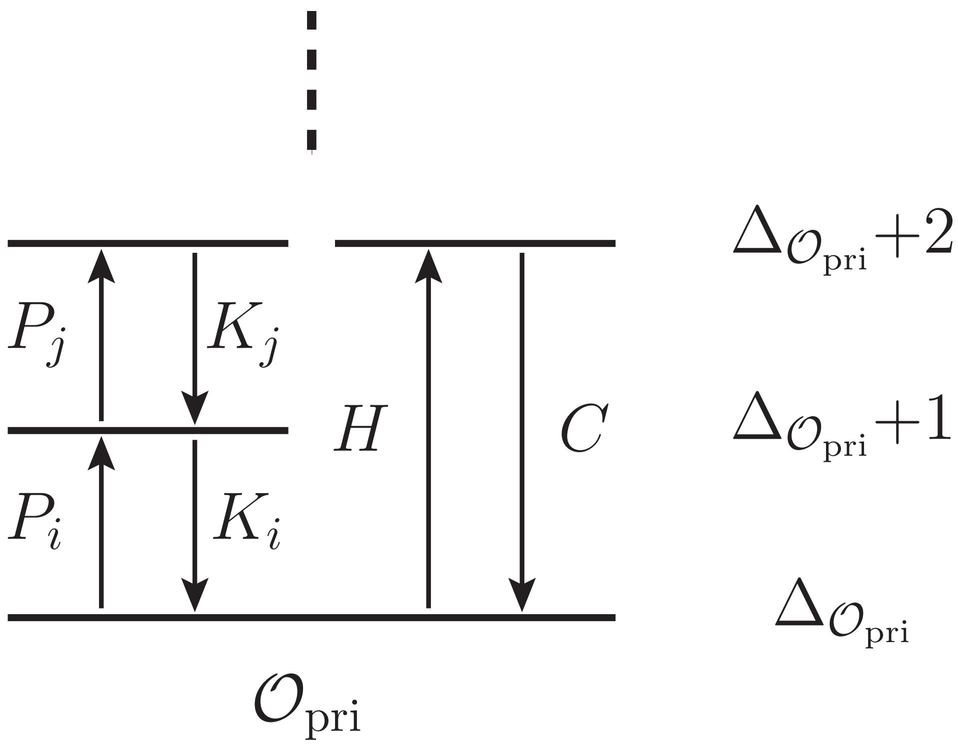
For example, $N_\psi = -1, \quad \Delta_\psi = 3/2$

Primary operators

$[D,\, P_i] = iP_i$ if dim[O]=Δ, then dim[P_i, O]=Δ+1

$$[D,\, H] = 2iH, \quad \dim[H,\, O] = \Delta + 2$$

$$[D,\, K_i] = -iK_i, \quad \dim[K_i,\, O] = \Delta - 1$$



O is primary operator if it cannot be lowered further

$$[K_i,\, O(0)] = [C,\, O(0)] = 0$$

Examples of primary operators

$\psi(x)$: $$[K_i\,\psi(0)] = \int d\mathbf{x}\, x_i [n(\mathbf{x}),\, \psi(0)] = -\int d\mathbf{x}\, \psi(\mathbf{x})\delta(\mathbf{x}) = 0$$

Homework: show that $\psi_\uparrow \partial_i \psi_\downarrow - \partial_i \psi_\uparrow \psi_\downarrow$

is a primary operator,

but not $\psi_\uparrow \partial_i \psi_\downarrow + \partial_i \psi_\uparrow \psi_\downarrow = \partial_i (\psi_\uparrow \psi_\downarrow)$

Operator-state correspondence

Consider operators made out of annihilation operators

Primary operator with dimension Δ	$\leftrightarrow$	Eigenstate in harmonic potential with energy ħΔω

Proof:

$$H_{\rm osc} = H + \omega^2 C \qquad C = \frac{1}{2}\int d\mathbf{x}\, \mathbf{x}^2 n$$

$$|\Psi_O\rangle \equiv e^{-H/\omega} O^\dagger(0)|0\rangle$$

$$\begin{aligned} H_\omega|\Psi_\mathcal{O}\rangle &= \left(H + \omega^2 C\right) e^{-H/\omega}\mathcal{O}^\dagger(0)|0\rangle \\ &= e^{-H/\omega}\left(\omega^2 C - i\omega D\right)\mathcal{O}^\dagger(0)|0\rangle = \omega\Delta_\mathcal{O}|\Psi_\mathcal{O}\rangle \end{aligned}$$

H,D,C form SO(2,1)

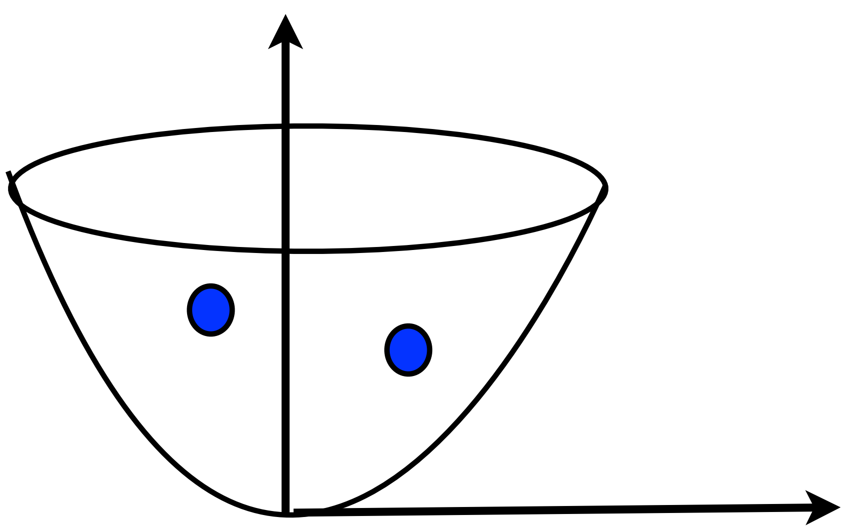
Operator-state correspondence: examples

$$\dim[\psi] = \frac{3}{2}$$

ground state of one particle in harmonic potential: E=3/2 ħω

Two particles

Ground state known exactly



$$\psi(\mathbf{x}, \mathbf{y}) = \frac{e^{-(x^2+y^2)/2}}{|\mathbf{x} - \mathbf{y}|}$$

E=2ħω

So the operator $\psi_\uparrow \psi_\downarrow$ has dimension 2

Naive dimension=3, anomalous dimension= -1?

Dimer operator

Consider a 2-body state characterized by a wavefunction $\Psi(x,y)$, call that state $|\Psi\rangle$

$$\langle 0|\psi_\uparrow(\mathbf{x})\psi_\downarrow(\mathbf{y})|\Psi\rangle = \Psi(\mathbf{x},\mathbf{y})$$

But recall the unitary boundary condition:

$$\Psi(\mathbf{x},\mathbf{y}) = \frac{C}{|\mathbf{x}-\mathbf{y}|} + \cdots$$

The operator $\psi_\uparrow(x)\psi_\downarrow(x)$ has infinite matrix elements

The properly defined two-body operator is

$$\phi(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{y}} 4\pi|\mathbf{x}-\mathbf{y}|\psi_\uparrow(\mathbf{x})\psi_\downarrow(\mathbf{y})$$

Matrix elements of ϕ are finite

Dimer in QFT

$$L = i\psi^\dagger \partial_t \phi - \frac{|\nabla\psi|^2}{2m} - (\psi_\uparrow^\dagger \psi_\downarrow^\dagger \phi + \text{h.c}) + c_0^{-1} \phi^\dagger \phi$$

=0 in dim reg

$$\langle \phi(x)\phi^\dagger(0)\rangle \sim \int \frac{dq_0, d\mathbf{q}}{(2\pi)^4} \frac{e^{iq\cdot x}}{\sqrt{\mathbf{q}^2/4m - q_0}} \sim \theta(t)\frac{1}{t^2}\exp\left(\frac{im\mathbf{x}^2}{t}\right)$$

OPE:

$$\psi_\uparrow(t,\mathbf{x})\psi_\downarrow(0) \sim \frac{1}{4\pi|\mathbf{x}|} f\left(\frac{t}{\mathbf{x}^2}\right)\phi(0)$$

Three-body operators

From the spectrum of 3 body in harmonic potential: lowest 3-body primary operators

$$\Delta_{l=1} = 4.27272 \qquad \Delta_{l=0} = 4.66622$$

4-body operator: lowest one has dimension ~ 5.0

Deforming the unitarity fermions:

$$L = L_{\rm CFT} + \frac{1}{4\pi a}\phi^*\phi + \# O^\dagger_{3,l=0} O_{3,l=0} + \#' O^\dagger_{3,l=1} O_{3,l=1}$$

suppressed by fractional powers of Λ

cf. Grießhammer

Comments on OPE

Product of two local operators expanded in sum over local operators

$$A(t,\mathbf{x})B(0,\mathbf{0}) = \sum_i |\mathbf{x}|^{\Delta_i-\Delta_A-\Delta_B} f_i\left(\frac{|\mathbf{x}|^2}{t}\right) O_i(0).$$

First applied to unit. fermions by Braaten and Platter

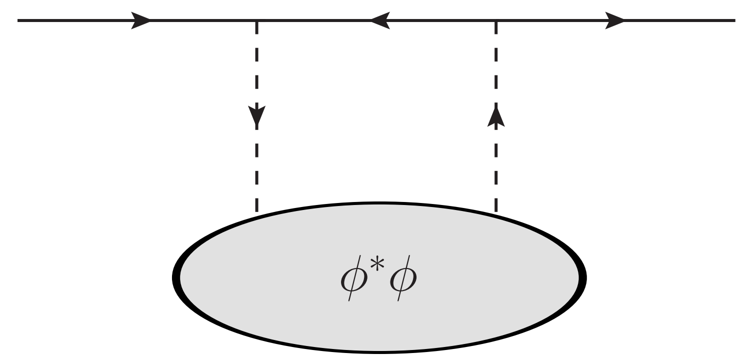
For high momentum (frequency) physics: only a first few operators with lowest dimensions matter

N=0 L=0 operators: n (Δ=3) and $\phi^\dagger\phi$ (Δ=4)

OPE coeffs: few-body calculations
expectation value can be taken in any state (e.g. finite μ)

a bridge between few- and many-body physics

OPE (II)



$$\int d\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}} \langle \psi_1^\dagger(0,\mathbf{0})\psi_1(t,\mathbf{x})\rangle = \frac{1}{q^4}\exp\left(i\frac{q^2}{2}t\right)\langle\phi^*\phi\rangle + \cdots$$

t=0: high-momentum tail of the distribution function

$$n_{\mathbf{q}} = \frac{\langle\phi^*\phi\rangle}{q^4},$$

Tan's parameter

OPE (III)

Dynamic structure factor S(q,ω) of unitarity fermions at high frequency, high momentum

$$S(\mathbf{q},\omega) = \sum_n \frac{1}{\omega^{\Delta_n - 1/2}} f_n\left(\frac{\mathbf{q}^2}{\omega}\right) \langle O_n^\dagger O_n \rangle$$

Leading inelastic contribution: from Tan's parameter

$$S(\mathbf{q},\omega) = \frac{\langle \phi^\dagger \phi \rangle}{\omega^{3/2}} f\left(\frac{\mathbf{q}^2}{\omega}\right)$$

f calculated: DTS, Thompson

Between 2 and 3 body thresholds: $2 < q^2/2\omega < 3$: leading contribution from 3-body operator

$$S(\mathbf{q},\omega) \sim \omega^{-3.77}$$

Holography

- Holography = mapping of a field theory into theory with gravity
- Most studied example: N=4 supersymmetric Yang-Mills theory
- Approximate models of strongly coupled quark-gluon plasma
- Models for unitarity fermions?

Schrödinger spacetime

$$ds^2 = \frac{-2dx^+dx^- + d\vec{x}^2 + dz^2}{z^2} - \frac{2(dx^+)^2}{z^4}$$

$$\begin{aligned}
P^i &: x^i \to x^i + a^i, \quad H: \; x^+ \to x^+ + a, \quad M: \; x^- \to x^- + a, \\
K^i &: x^i \to x^i - a^i x^+, \quad x^- \to x^- - a^i x^i, \\
D &: x^i \to (1-a)x^i, \quad z \to (1-a)z, \quad x^+ \to (1-a)^2 x^+, \quad x^- \to x^-, \\
C &: z \to (1 - a x^+)z, \quad x^i \to (1 - a x^+)x^i, \quad x^+ \to (1 - a x^+)x^+, \\
&\qquad x^- \to x^- - \frac{a}{2}(x^i x^i + z^2).
\end{aligned}$$

D.T.S, arXiv:0804.3972,

Balasubramanian, McGreevy arXiv:0804.4053

Conclusions

- Fermions at unitarity present a challenging theoretical problem
- Direct link between many-body and few-body physics comes from operator product expansion
- Some ideas about holography, but the approach still seems underdeveloped.