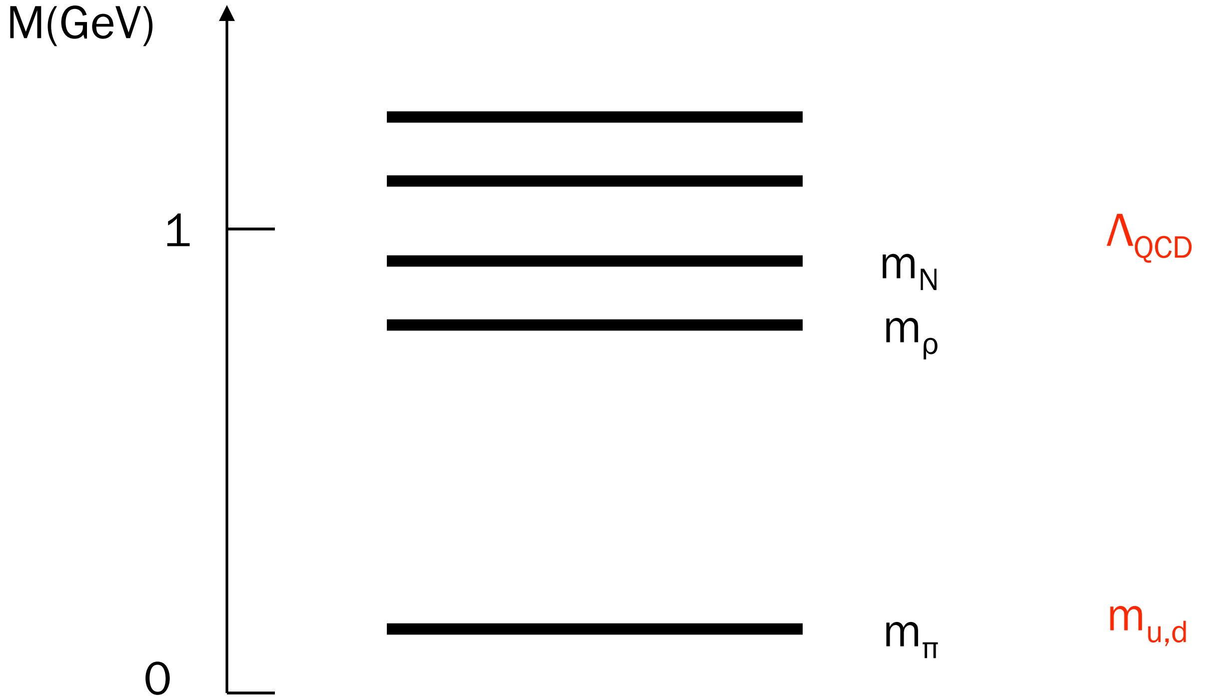
Lecture 2: Chiral Perturbation Theory

Mass Scales

QCD in the light quark (up & down) sector (QCD-light) has two mass scales



Effective Field Theory

- **In a generic physical system, there are often many scales involved. However, for a specific problem under consideration, it may depends on physics only on a particular scale.**
- **One can "integrate out" physics at other scales and focus on the dynamics on the degrees of freedom relevant to that scale: Effective (Field) Theory**
- **Many example:**
  - Fluid Dynamics
  - Multiple expansion in Electrodynamics
  - Nuclear Physics
  - ....

Chiral Symmetry of QCD-light

- When quarks are massless, N_f flavor of QCD lagrangian has $U_L(N_f)$ x $U_R(N_f)$ chiral symmetry.
- Each quark has a left-handed and right-handed components,

$$\psi_{Lf} = \frac{1}{2}(1-\gamma_5)\psi_f \qquad \psi_{Rf} = \frac{1}{2}(1+\gamma_5)\psi_f .$$

- The left and right-handed fields do not couple to each others in the massless limit. Each fields can rotate independently producing a symmetry group $U_L(N_f) \times U_R(N_f)$

$$\mathcal{L}_q = \mathcal{L}_q(\psi_L) + \mathcal{L}_q(\psi_R) , \qquad \begin{pmatrix} u'_{L,R} \\ d'_{L,R} \end{pmatrix} = U_{L,R} \begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix}$$

The Folklore

- $U_L(N_f) \times U_R(N_f)$ contains two U(1) symmetries: vector and axial: Vector U(1) is related to baryon number, and the axial U(1) is broken by anomaly.
- The anomaly is a phenomenon that a classical symmetry is broken by quantum fluctuations, and was first discovered by Adler, Bell, and Jackiw.
- The remaining chiral symmetry $SU_L(N_f) \times SU_R(N_f)$ is broken **spontaneously** to $SU(N_f)$ flavor symmetry (isospin) discussed in the previous lecture.
- SSB: spontaneous symmetry breaking.

A Bit Group Theory

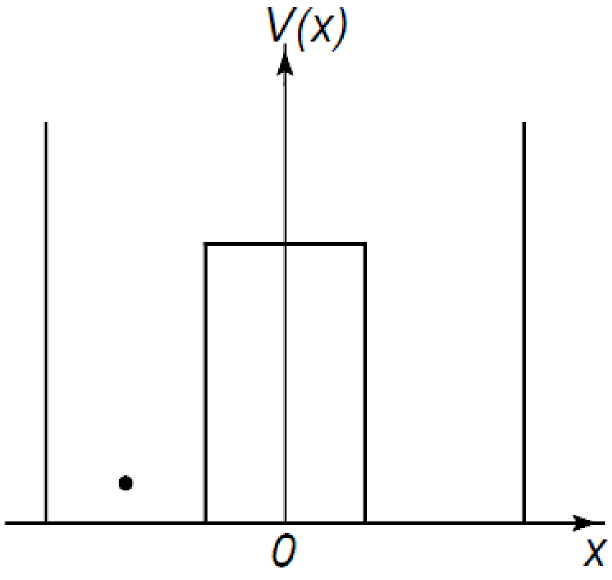
- The representation of SU(2) group [angular momentum algebra] contains dimensions, 1, 2, 3… (2j+1),…
- Therefore the representation of the chiral group $SU_L(2)\times SU_R(2)$ can be labeled by $(2j_1+1, 2j_2+1)$.
- The left-handed quark field is (2,1) and the right-handed quark field is (1,2).
- Isospin representations comes from adding the two reps.
- The quark mass terms $H_1 = m_u\bar{u}u + m_d\bar{d}d$

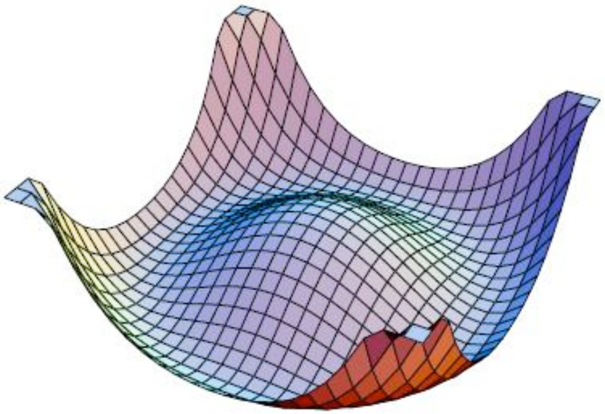
  can be decomposed into $H_1 = m_u(\bar{u}_L u_R + \bar{u}_R u_L) + m_d(\bar{d}_L d_R + \bar{d}_R d_L)$.

  it is a (2-bar, 2) + (2, 2-bar), not invariant under chiral symmetry

Spontaneous Symmetry Breaking

- A simple example is a particle moving in a double well.
- When the mid-barrier is finite, the ground state is always symmetric in x –› -x.
- However, when the height is going to infinity, the ground state is degenerate, and the physical ground state is for the particle in either wells, a broken symmetry state.
- Another example of SSB is spontaneous magnetization of a piece of magnet.





Nambu-Goldstone Theorem

- In the case of the SSB of a continuous symmetry, there are massless Goldstone bosons produced as result. This is because everywhere in the space, one can choose a different vacuum (vacuum degeneracy), there is no energy difference between the different choices.
- Pion would have been the massless Goldstone boson associated with the chiral symmetry breaking of $SU_L(2)\times SU_R(2)$, if the quark masses were zero.
- The pion interactions must be derivative-coupled because in the long wavelength limit, the interactions vanish, because the long-wavelength pion approaches the vacuum.

Order Parameter

- When SSB happens, there is an order parameter which characterizes the symmetry breaking.
- The physical vacuum no longer invariant under chiral symmetry, rather, it is a sum of chiral reps,

$$|\text{vac}\rangle >= |(1,1)\rangle + |(2,2)\rangle + |(3,3)\rangle + \ldots$$

  the chiral representation must have isospin 0, so $j_1=j_2$.
- Therefore, there is a non-zero chiral consendate in the physical vacuum, which characterizes the scale at which SSB happens

$$\langle 0|\bar{u}u + \bar{d}d|0\rangle$$

Non-linear realization of chiral symmetry

- Easiest way to see that pions are derivatively coupled is to introduce a U fields that transforms as (2,2) of the chiral group.

  U → LUR^{-1}

  which contains the Goldstone boson field.
- Construct lagrangian that are invariant under the chiral transformation.
- After SSB, U is related to the pion field,

$$U = \sigma e^{i\vec{\pi}^a(x)\cdot\tau^a/f_\pi}$$

Or we simply write this U= σΣ, here Σ is a non-linear realization of chiral symmetry.

Power counting

- When the energy of the pion is low, derivatives are small compared to the scale of SSB. Therefore, one can make expansion in ∂/f_π
- This expansion is called the chiral expansion.
- Taking into account the non-zero pion mass, m_π/f_π is another small expansion parameter.
- Chiral perturbation theory (ChiPT) carries out systematic expansion in these small parameters. Since the theory uses symmetry and SSB, test of ChiPT is usually considered as a test of QCD itself. [“If ChiPT does not work, QCD is in trouble.”]

Pion Mass

- Pion is massless in the chiral limit. Therefore, its non-zero mass must come from the non-zero quark mass.
- One can show,

$$m_\pi^2 = -(m_u + m_d)\langle 0|\bar{u}u + \bar{d}d|0\rangle / f_\pi^2$$

which is linear in quark mass and also related to the chiral condensate!

Lagrangian for Pion

The simplest lagrangian for pure pion involves the kinetic energies and pion mass, second-order in small parameter

$$L = \frac{f_\pi^{\ 2}}{4}\mathrm{Tr}\left[\partial_\mu \Sigma \partial^\mu \Sigma^+\right] + \frac{f_\pi^{\ 2}}{4} m_\pi^{\ 2}\mathrm{Tr}\left[\Sigma + \Sigma^+\right] + O(m_\pi^{\ 4})$$

the dependence in the pion mass is analytical in sense that it is a Taylor expansion.

Higher-order term can also be written down, involving more unknown constants, called chiral constants

$$\begin{aligned} &L_4 \mathrm{Tr}(D^\mu \Sigma^\dagger D_\mu \Sigma)\mathrm{Tr}(\chi^\dagger \Sigma + \chi \Sigma^\dagger) + L_5 \mathrm{Tr}(D^\mu \Sigma^\dagger D_\mu \Sigma)(\chi^\dagger \Sigma + \chi \Sigma^\dagger) \\ &+ L_6 (\mathrm{Tr}(\chi^\dagger \Sigma + \chi \Sigma^\dagger))^2 + L_7 (\mathrm{Tr}(\chi^\dagger \Sigma - \chi \Sigma^\dagger))^2 \\ &+ L_8 \mathrm{Tr}(\chi^\dagger \Sigma \chi^\dagger \Sigma + \chi \Sigma^\dagger \chi \Sigma^\dagger) + H_2 \mathrm{Tr}(\chi^\dagger \chi) \end{aligned}$$

Pion-pion Scattering

- Expand the pion lagrangian to the first non-trivial order

$$\mathcal{L}_{\pi\pi}^{(2)} = \frac{1}{2}(\partial_\mu \vec{\pi})^2 + \frac{1}{6f_\pi^2}[(\partial_\mu \vec{\pi} \cdot \vec{\pi})^2 - \vec{\pi}^2(\partial_\mu \vec{\pi})^2] + \ldots$$

There is no unknown parameter!

- Taking into account the pion mass effects as well,

$$\mathcal{M} = -f_\pi^{-2}\left(\delta_{ab}\delta_{cd}(s - m_\pi^2) + \delta_{ac}\delta_{bd}(t - m_\pi^2) + \delta_{ad}\delta_{bc}(u - m_\pi^2)\right)$$

- Scattering length in isospin 0 and 2 sectors,

$$a_0 = 7m_\pi/32\pi f_\pi^2 = 0.16 m_\pi^{-1}$$

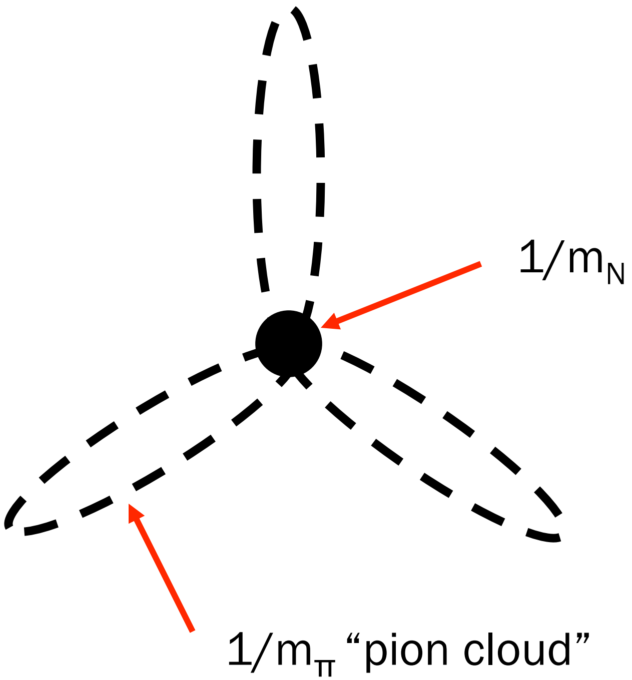
$$a_2 = -2m_\pi/32\pi f_\pi^2 = -0.046 m_\pi^{-1}$$

Experimentally,

$$a_0 = .26 \pm 0.5$$

$$a_2 = -0.028 \pm 0.012$$

Chiral Physics in the Nucleon



What's calculable and what's not?

- **Since only the long distance part of nucleon physics is related to the pion and is calculable using the ChiPT, the short distance physics are parameterized by the so-called low-energy constants. There are large number of such low-energy constants.**
- **The predictive power of ChiPT comes from distinctive contributions of the pion, non-analytic contributions from the pion mass.**

Singular Contribution and IR divergence

- **Loop calculations depend on the pion propagator:**

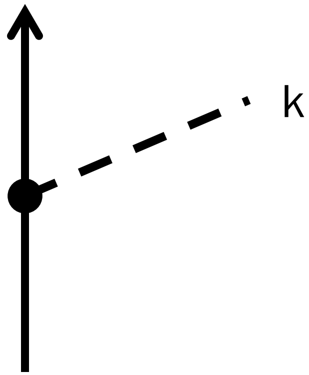
$$\frac{i}{k^2 - m_\pi^{\ 2} + i\varepsilon}$$

with loop momentum-*k* to be integrated over. The integrations can generate non-analytic dependence on m_π^2

What are they? $\mathbf{1/m_\pi^n}$ (n>0), $\mathbf{m_\pi^{2n+1}}$, $\mathbf{ln(m_\pi)}$, …

- **There dependence usually comes from IR divergences.**
- **Dependence on m_π^2 in the counter terms are analytic because they are treated in perturbative expansion.**

Pion-Nucleon Coupling



$$\frac{g_A}{f_\pi} \vec{\sigma} \cdot \vec{k} \tau^a$$

g_A: neutron decay constant, dimension 0
f_π: pion decay constant, dimension 1

HBChPT vs. Relativisitc ChPT

Heavy-Baryon Chiral Perturbation Theory

Get rid of the hadron mass scale, m_N ->∞.

Physics at scale m_N is not really calculable in chiral perturbation theory, should be included in the counter terms.

Relativistic Chiral Perturbation Theory

Contain partial high-order contributions.

Better or Worse? Don’t know. They provide some idea on the size of higher-order corrections.

The Nucleon Mass

- At leading order (one-loop), two powers of

  $1/4\pi f_\pi$
- Since the contribution must have a dimension of mass

$$\delta m_N \sim \frac{m_\pi^{\;3}}{(4\pi f_\pi)^2}$$

Since this is nonanalytic, the coefficient is calculable! ($3/\pi 2$). It contributes –15 MeV to the nucleon mass.

There is a m_π^2 contribution, proportional to the σ-term.

Isovector charge radius

- $\langle r^2 \rangle$ has a mass-dimension –2.
- Leading pion-loop has a factor of $1/(4\pi f_\pi)^2$
- Therefore, the chiral contribution goes like

$$\frac{1}{(4\pi f_\pi)^2}\ln(m_\pi^{\ 2}/\Lambda^2)$$

which diverges as $m_\pi \to 0$. (coefficient $(5g_A^{\ 2}+1)$)

just like the charge radius of the electron in QED!

- Isoscalar charge radius is regular as $m_\pi \to 0$.
- **Small neutron charge radius is an accident!**

Isovector magnetic moment

- Magnetic moment has a mass dimension –1.
- Leading pion-loop has a factor of $1/(4\pi f_\pi)^2$
- Thus the nonanalytical chiral contribution,

$$\frac{m_\pi}{(4\pi f_\pi)^2}$$

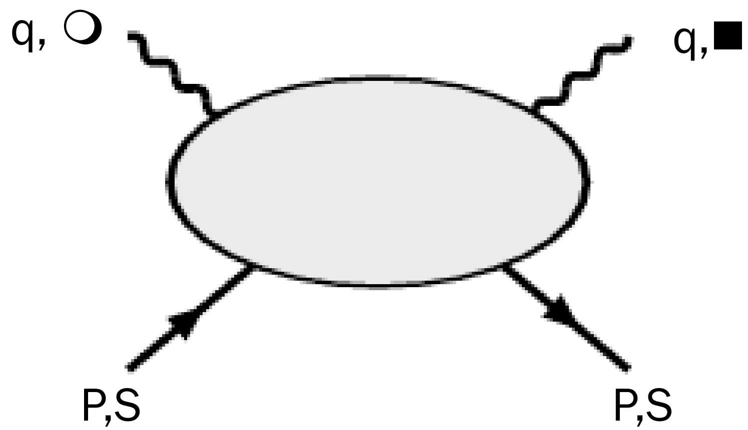
- Coefficient is $-2\pi g_A^2$

  A significant contribution at physical pion mass.

Low-energy Scattering off the Nucleon

- **Compton Scattering and Sum Rules**
  - Real Photon
  - Virtual Photon
  - Doubly Virtual Photon
- **Pion-photo and electroproduction**
- **Pion scattering**
- **….**

Spin-Dependent Forward Compton Scattering



- **Two Compton amplitudes:**

$$T^{[\mu\nu]}(P,q,S) = -i\epsilon^{\mu\nu\alpha\beta} q_\alpha \left[S_\beta S_1(\nu, Q^2) + (M\nu\, S_\beta - S\cdot q\, P_\beta)\, S_2(\nu, Q^2) \right] .$$

S_1 and S_2 at low energy can be calculated in CHIPT

Dispersion Sum Rules

- Unsubtracted dispersion relations

$$S_1(\nu, Q^2) = 4\int_{Q^2/2M}^{\infty} \frac{d\nu'\nu' G_1(\nu', Q^2)}{\nu'^2 - \nu^2} ,$$

G_1 is the spin-dependent structure function.

- Expand at small ■,

$$S_1(\nu, Q^2) = \sum_{n=0,2,4,\ldots} \nu^n S_1^{(n)}(Q^2) ,$$

- Dispersion sum rules valid at all Q^2

$$S_1^{(n)}(Q^2) = 4\int_{Q^2/2M}^{\infty} \frac{d\nu}{\nu^{n+1}} G_1(\nu, Q^2) \quad (n = 0, 2, 4, \ldots) ,$$

X.Ji & J. Osborne, JPG27, 127 (2001)

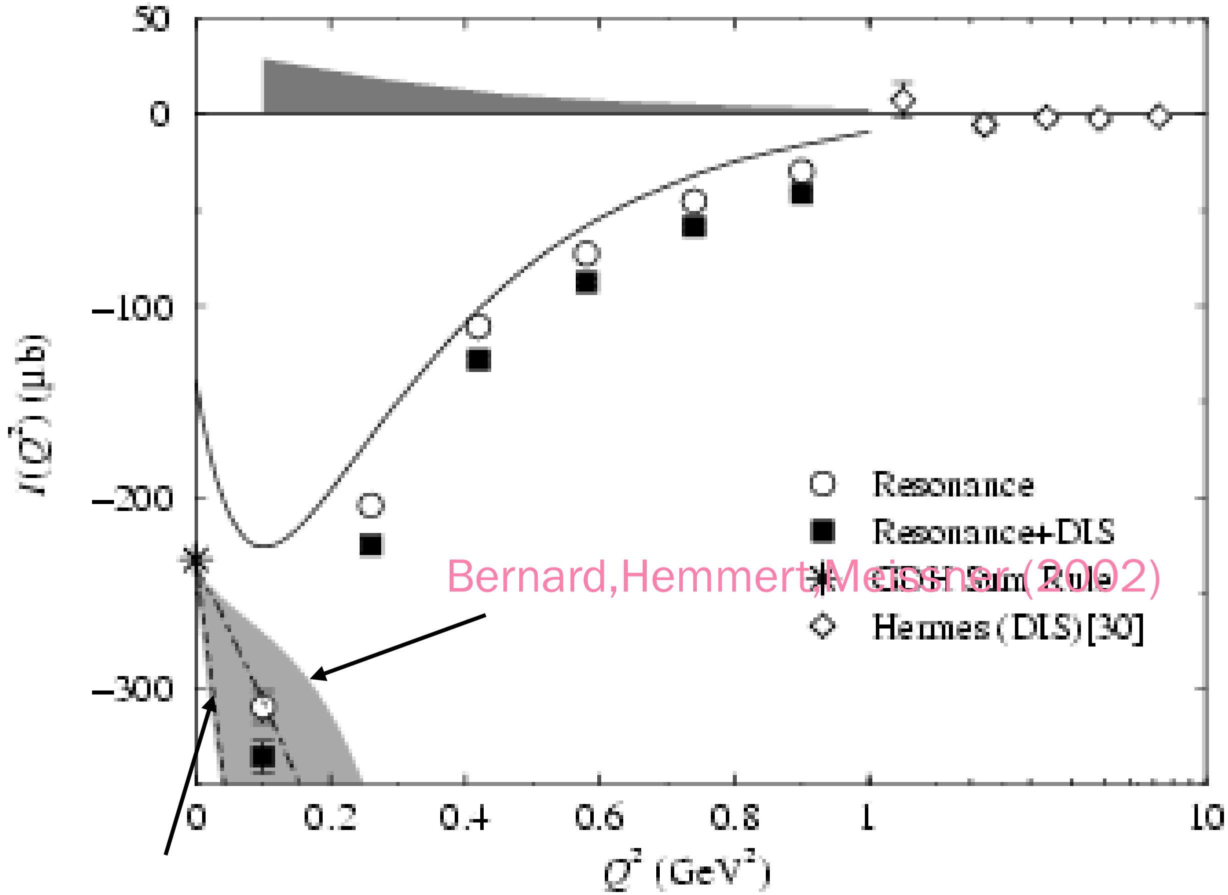
The First G_1 Sum Rule

$$S_1(0,Q^2) = 4\int_{Q^2/2M}^{\infty} \frac{d\nu}{\nu} G_1(\nu,Q^2) \,.$$

- **At low-Q^2, $S_1(0,Q^2)$ can be calculated in CHIPT**
  - At $O(p^3)$, $S_1(0,Q^2)$ is zero
  - At $O(p^4)$: Ji, Kao, Osborne, PLB472,1(2000)

$$\overline{S_1}^{O(p^4)}(0,Q^2) = \frac{g_A^2 \pi m_\pi}{8(4\pi f_\pi)^2 M}\Big[-2(5+6\kappa_V+(1+6\kappa_S)\tau^3) + \left(4\left(5+6\kappa_V+(1+6\kappa_S)\tau^3\right) + \frac{Q^2}{m_\pi^2}\left(3+6\kappa_V+(3+10\kappa_S)\tau^3\right)\right) \times \sqrt{\frac{m_\pi^2}{Q^2}}\sin^{-1}\sqrt{\frac{Q^2}{4m_\pi^2+Q^2}}\Big],$$

Data: M. Amarian et.al. PRL89, 242301 (2002)

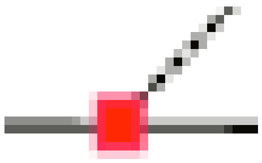


Bernard,Hemmert,Meissner (2002)

Ji,Kao,Osborne (2000)

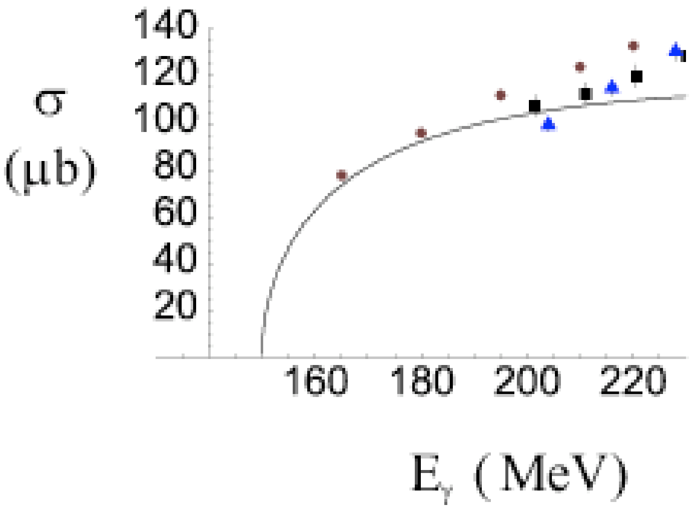
Parity-Violating Photo-Pion Production

- Can be used to measure the **parity-violating pion-nucleon coupling**



$$\mathcal{L}^{PV} = -ih^{(1)}_{\pi NN}\pi^+ p^\dagger n + h.c. + \cdots ,$$

- Calibration:



Low-energy theorem

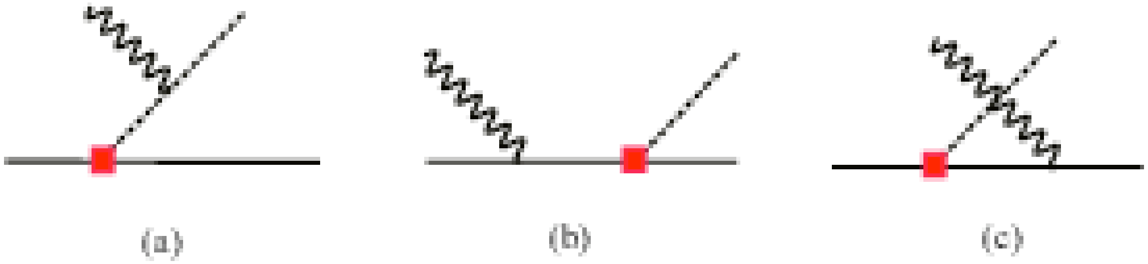


FIG. 2. Feynman diagrams contributing to the parity-violating amplitudes at LO ($\mathcal{O}(1)$) and NLO ($\mathcal{O}(p)$) in $\vec{\gamma} p \to \pi^+ n$.

$$A_\gamma(\omega_{th}, \theta) = \frac{\sqrt{2} f_\pi (\mu_p - \mu_n)}{g_A m_N} h^{(1)}_{\pi NN} ,$$

$\sim 2 \times 10^{-7}$

J.W.Chen & X.Ji, PRL86,4239 (2000)