Hadron-Hadron Scattering in Lattice QCD

Aaron Torok

University of New Hampshire

Overview

- Energy Eigenvalues of 2 Particles in a Box
- LQCD Calculation
- Correlation Functions and Fitting
- Jackknife
- Some Effective Mass Plots
- Extrapolations using χ-PT

Energy Eigenvalues

The exact energy eigenvalue equation for E_n and its deviation from the sum of the rest masses of the particle, ΔE_n, are related to the center-of-mass momentum p_n by (for ex. pions)

$$\Delta E_n \quad \equiv \quad E_n \; - \; 2m_\pi \; = \; 2\sqrt{\; p_n^2 \; + \; m_\pi^2 \;} \; - \; 2m_\pi$$

Energy levels occur at momenta $\mathbf{p} = 2\pi\mathbf{j}/L$, corresponding to single-particle modes in a cubic volume. The Luscher formula then relates the phase shift to the momenta:

$$p \cot \delta(p) \; = \; \frac{1}{\pi L} \, \mathbf{S} \left(\frac{pL}{2\pi} \right)$$

$$\mathbf{S} \left(\frac{pL}{2\pi} \right) \; \equiv \; \sum_{\mathbf{j}}^{\Lambda_j} \frac{1}{|\mathbf{j}|^2 - \left(\frac{pL}{2\pi} \right)^2} \; - \; 4\pi\Lambda_j$$

we can use the effective range expansion for $pcot\delta(p)$, keeping only the first term $(-1/a)$ since $p \to 0$

Energy Eigenvalues and Scattering Length

The scattering length can be expressed in terms of known constants, and quantities we can measure on the lattice;

$$\Delta E_0 \quad = \quad -\frac{4\pi a}{ML^3}\left[1 \, + \, c_1 \frac{a}{L} \, + \, c_2 \left(\frac{a}{L}\right)^2\right] \, + \, \mathcal{O}\left(\frac{1}{L^6}\right)$$

where the constants, $c_1\,, c_2$ contain infinite sums, and a regulator Λ, which have to be evaluated numerically (see S.R. Beane, P.F.Bedaque, A. Parreno, M.J. Savage, hep-lat/0312004)

Using the above expression, we can solve for the scattering length since we can fit both masses and ΔE, from our lattice data.

This expression is obtained from the exact equation for $\mathbf{S}\left(\frac{pL}{2\pi}\right)$.

LQCD Calculation Configurations

In the non-perturbative regime, Lattice QCD is one technique that can be used to calculate observables. Some representative gauge configurations are shown (can be downloaded from http://qcd.nersc.gov/):

Coarse Config Set	Dimensions	bm_l	bm_s	bm_{dwf}	m_π (extracted)	# configs	# sources
2064f21b676m007m050	$20^3 \times 64$	0.007	0.05	0.0081	294 MeV	468	16
2064f21b676m010m050	$20^3 \times 64$	0.010	0.05	0.0138	348 MeV	658	20
2064f21b679m020m050	$20^3 \times 64$	0.020	0.05	0.0313	484 MeV	486	24
2064f21b681m030m050	$20^3 \times 64$	0.030	0.05	0.0474	565 MeV	564	8

$$b = 0.125 \text{ fm} \qquad L = 2.5 \text{ fm}$$

- We use the Chroma software system for LQCD (see Edwards, Joo; hep-lat/0409003), which is based on QDP++. The software uses an XML file for input parameters
- the simulation returns $q\overline{q}$ and qqq objects with uncontracted spin and color indices.
- use contraction code to obtain the correlation function of interest, these are the correlation amplitudes that we fit to.

π^+ and $\pi^+\pi^+$ Correlation Functions

The correlation functions are computed as follows:

$$C_{\pi+}(t) \quad = \quad \sum_{\mathbf{x}} \langle \pi^-(t,\mathbf{x})\ \pi^+(0,\mathbf{0})\rangle$$

where the summation over $\mathbf{x}$ corresponds to summing over all the spatial lattice sites, thereby projecting onto the momentum $\mathbf{p}=\mathbf{0}$ state. A $\pi^+\pi^+$ correlation function that projects onto the s-wave state in the continuum limit is

$$C_{\pi+\pi+}(p,t) \quad = \quad \sum_{|\mathbf{p}|=p} \sum_{\mathbf{x},\mathbf{y}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \langle \pi^-(t,\mathbf{x})\ \pi^-(t,\mathbf{y})\ \pi^+(0,\mathbf{0})\ \pi^+(0,\mathbf{0})\rangle$$

And the following ratio yields the energy:

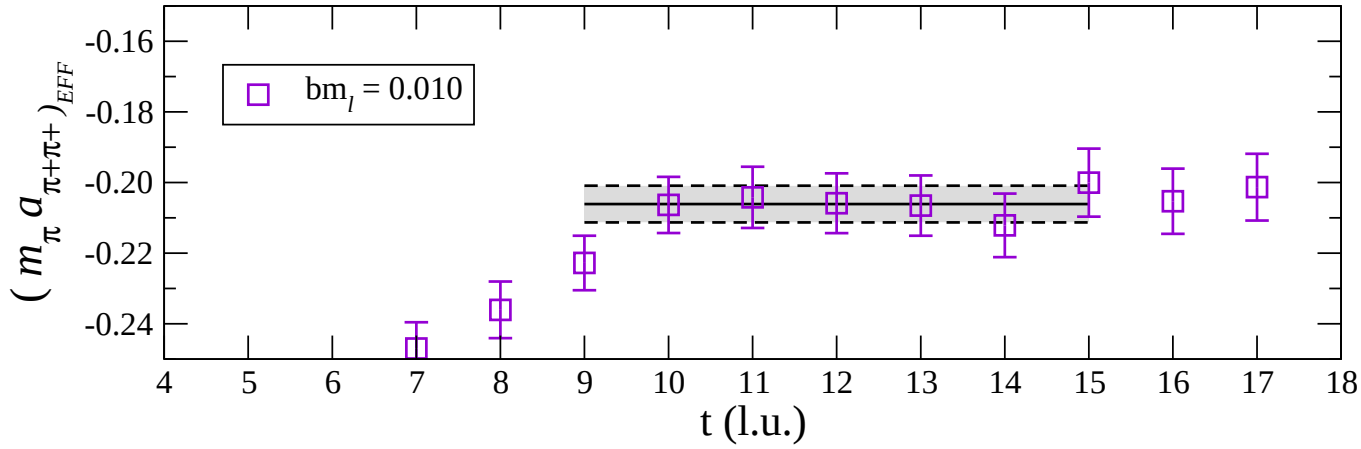
$$G_{\pi+\pi+}(p,t) \quad \equiv \quad \frac{C_{\pi+\pi+}(p,t)}{C_{\pi+}(t)C_{\pi+}(t)} \ \rightarrow \ \sum_{n=0}^{\infty} \mathcal{A}_n\ e^{-\Delta E_n\ t}$$

Fitting the Correlation Function Data

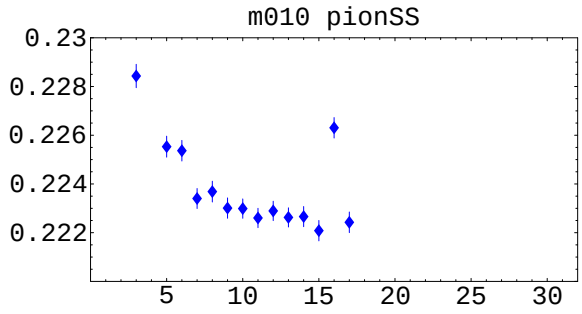
In order to determine an appropriate range to fit the correlation function data to the function Ae^{-mt}, where m could be either the particle mass, or ΔE, effective plots are useful. These plots are constructed by taking the ratio:

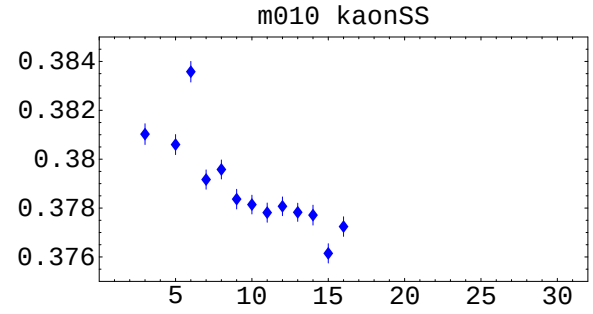
$$Q_{effective}(t) = \log\left(\frac{C(t)}{C(t+1)}\right)$$

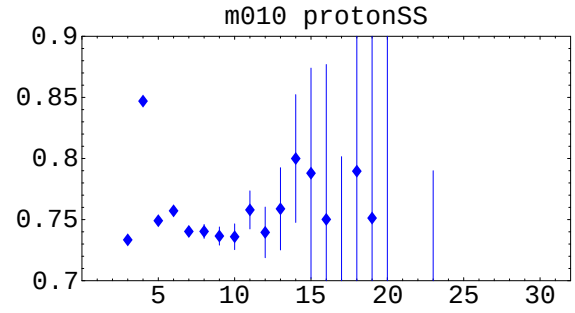
where $Q_{effective}(t)$ could be the mass, or another quantity, and the $C(t)'s$ are the correlation functions at adjacent time slices.

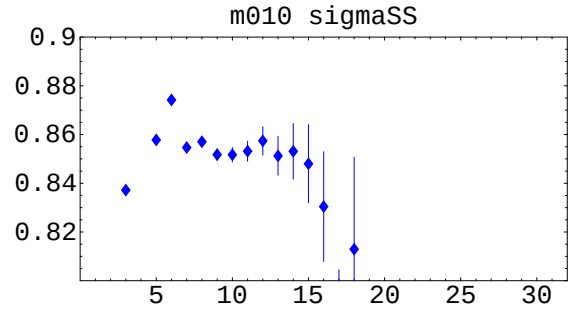


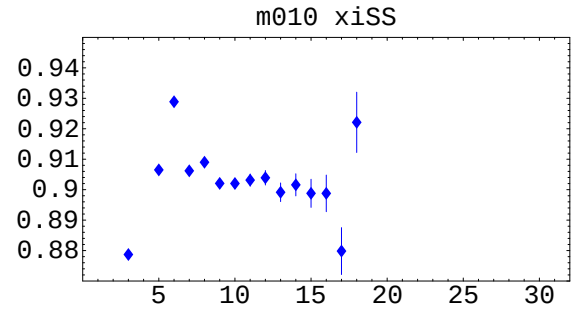
Some Effective Mass Plots











Jackknife

Once we have the numbers from the correlators, we average over the number of gauge configurations using the jackknife method

$$\alpha_i = [\alpha_1, \alpha_2, \cdots, \alpha_N]$$

$$\alpha_i^{jackknife} = \frac{1}{N-1}\left[\sum_{i=1}^{N}\alpha_i - \alpha_1, \sum_{i=1}^{N}\alpha_i - \alpha_2, \ldots, \sum_{i=1}^{N}\alpha_i - \alpha_N\right]$$

time	config 1	config 2	...	config N
0	0.00012	0.00013	...	0.00012
1	0.00007	0.00006	...	0.00004
⋮	⋮	⋮	⋮	⋮
31	0.00009	0.00008	...	0.00007

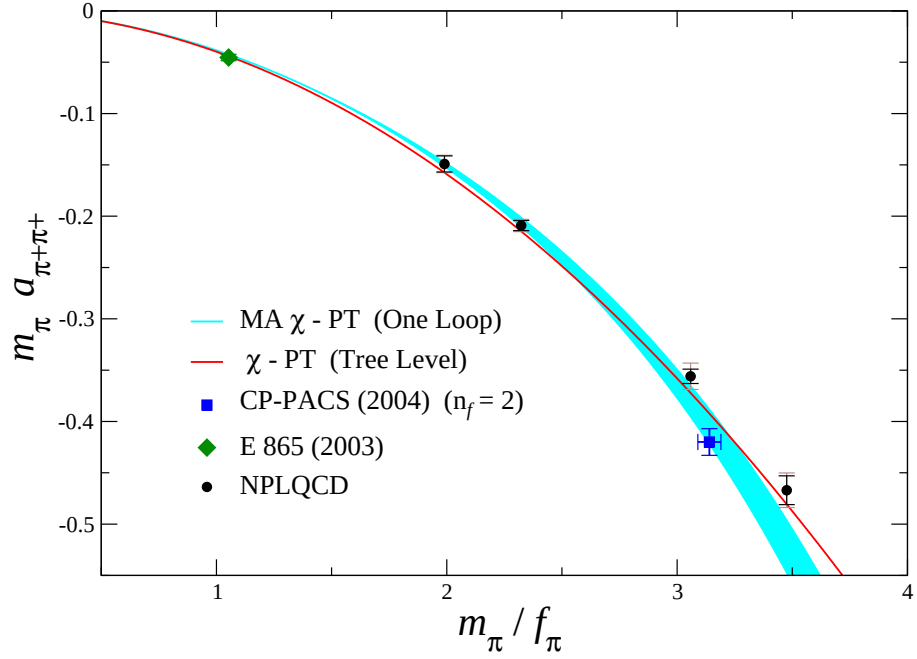
The reason for jackknife is that the measurements at different coordinates (x, y) are not statistically independent (see DeGrand & DeTar, "Lattice Methods for Quantum Chromodynamics").

χ-PT Extrapolation

So far, we could extract m_π, ΔE and a, at the quark masses: 0.007,0.010,0.020,and 0.030. At leading order, the expression for $m_\pi a_{\pi\pi}^{I=2}$ from χ-PT is:

$$m_\pi a_{\pi\pi}^{I=2} = -\frac{m_\pi^2}{8\pi f_\pi^2} + \ldots$$

the other parameter we need is f_π, and in order to obtain this, we use the constant in front of the exponential in the fit, comparing different kinds of sources generated on the lattice. With f_π, then each data point can be plotted:



Remarks

the paper describing $I = 2\pi\pi$ in more detail can be found at: arXiv:0706.3026

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