Nuclear & Particle Physics of Compact Stars

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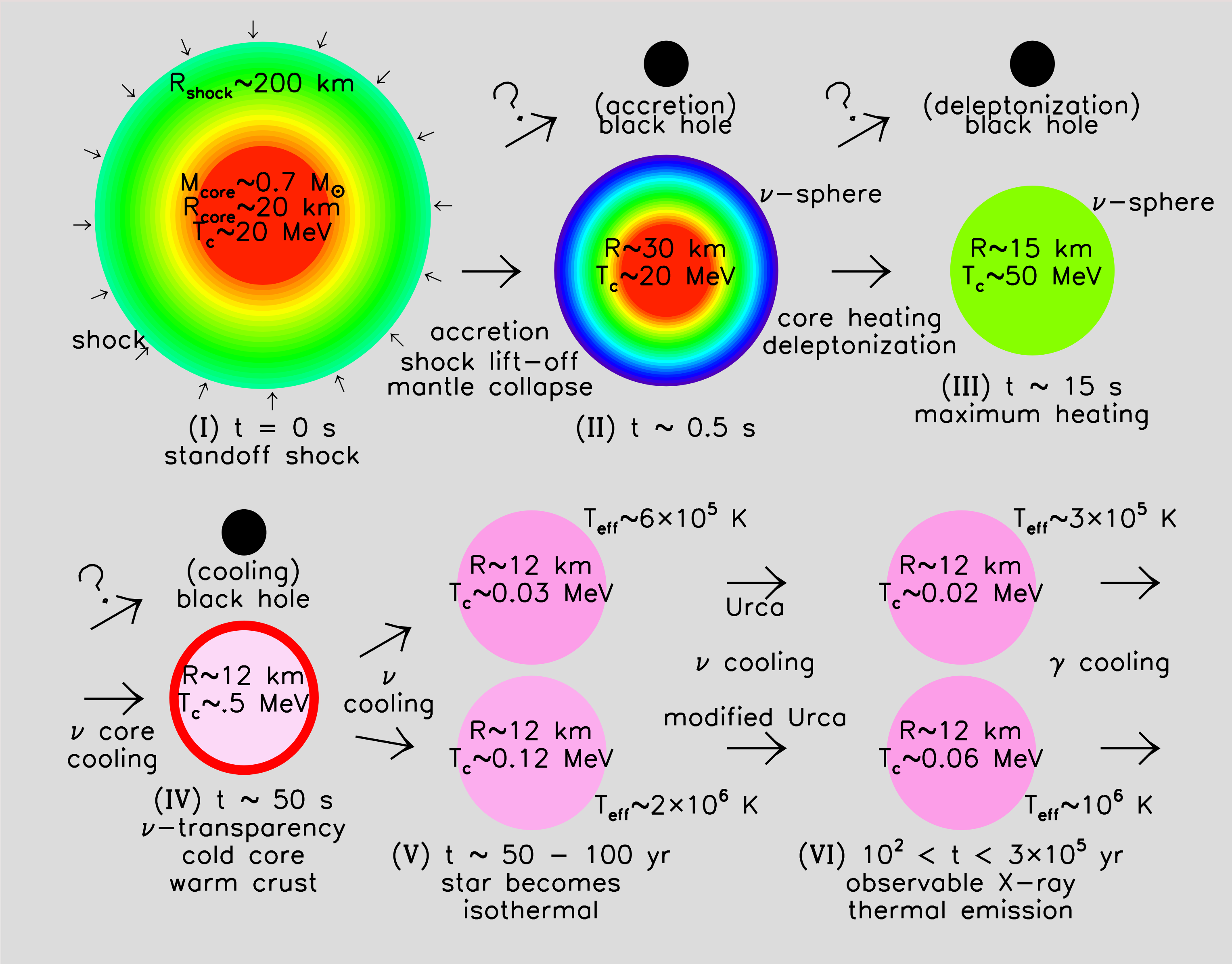
How Neutron Stars are Formed



Lattimer & Prakash , Science **304**, 536 (2004).

Equations of Stellar Structure-I

- In hydrostatic equilibrium, the structure of a spherically symmetric neutron star from the Tolman-Oppenheimer-Volkov (TOV) equations:

$$\frac{dM(r)}{dr} = 4\pi r^2 \epsilon(r)$$

$$\frac{dP(r)}{dr} = -\frac{GM(r)\epsilon(r)}{c^2 r^2} \frac{\left[1 + \frac{P(r)}{\epsilon(r)}\right]\left[1 + \frac{4\pi r^3 P(r)}{M(r)c^2}\right]}{\left[1 - \frac{2GM(r)}{c^2 r}\right]}$$

- G := Gravitational constant
- P := Pressure
- ϵ := Energy density
- $M(r)$:= Enclosed gravitational mass
- $R_s = 2GM/c^2$:= Schwarzschild radius

Equations of Stellar Structure-II

- The gravitational and baryon masses of the star:

$$M_G c^2 = \int_0^R dr\, 4\pi r^2\, \epsilon(r)$$

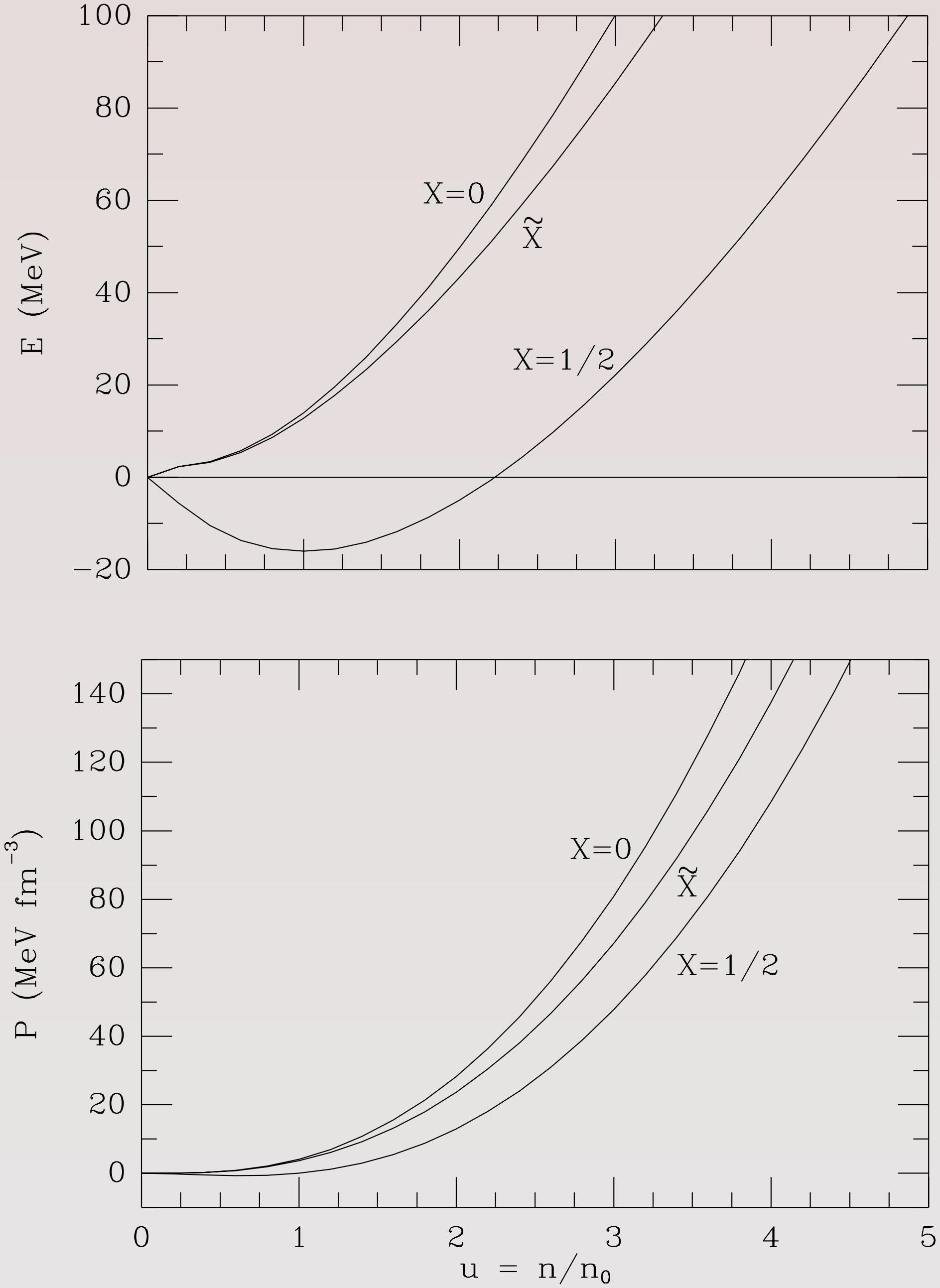
$$M_A c^2 = m_A \int_0^R dr\, 4\pi r^2\, \frac{n(r)}{\left[1 - \frac{2GM(r)}{c^2 r}\right]^{1/2}}$$

- m_A := Baryonic mass
- $n(r)$:= Baryon number density
- The binding energy of the star $B.E. = (M_A - M_G)c^2$.

To determine star structure :

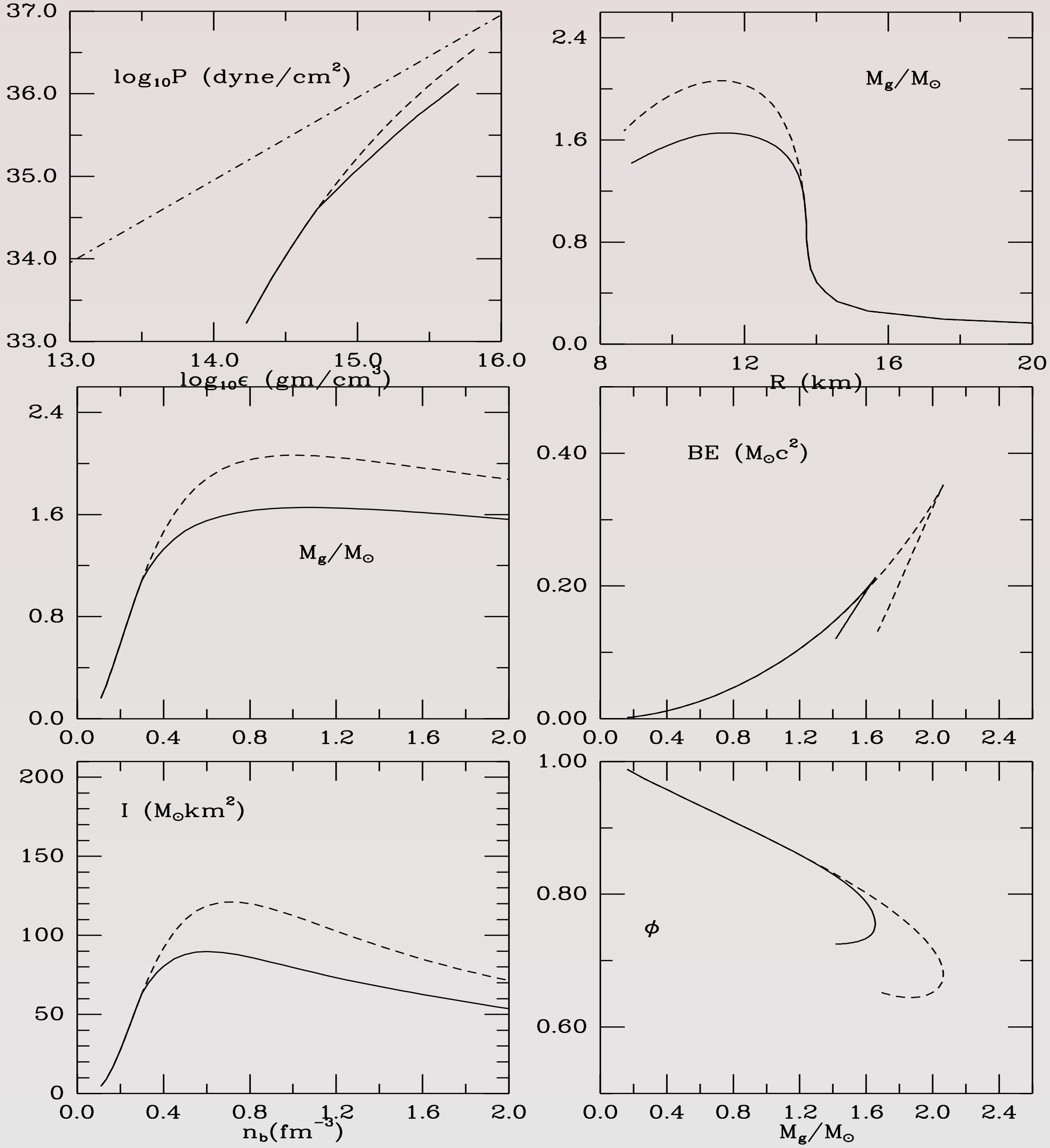
- Specify equation of state, $P = P(\epsilon)$
- Choose a central pressure $P_c = P(\epsilon_c)$ at $r = 0$
- Integrate the 2 DE's out to surface $r = R$, where $P(r = R) = 0$.

Nucleonic Equation of State



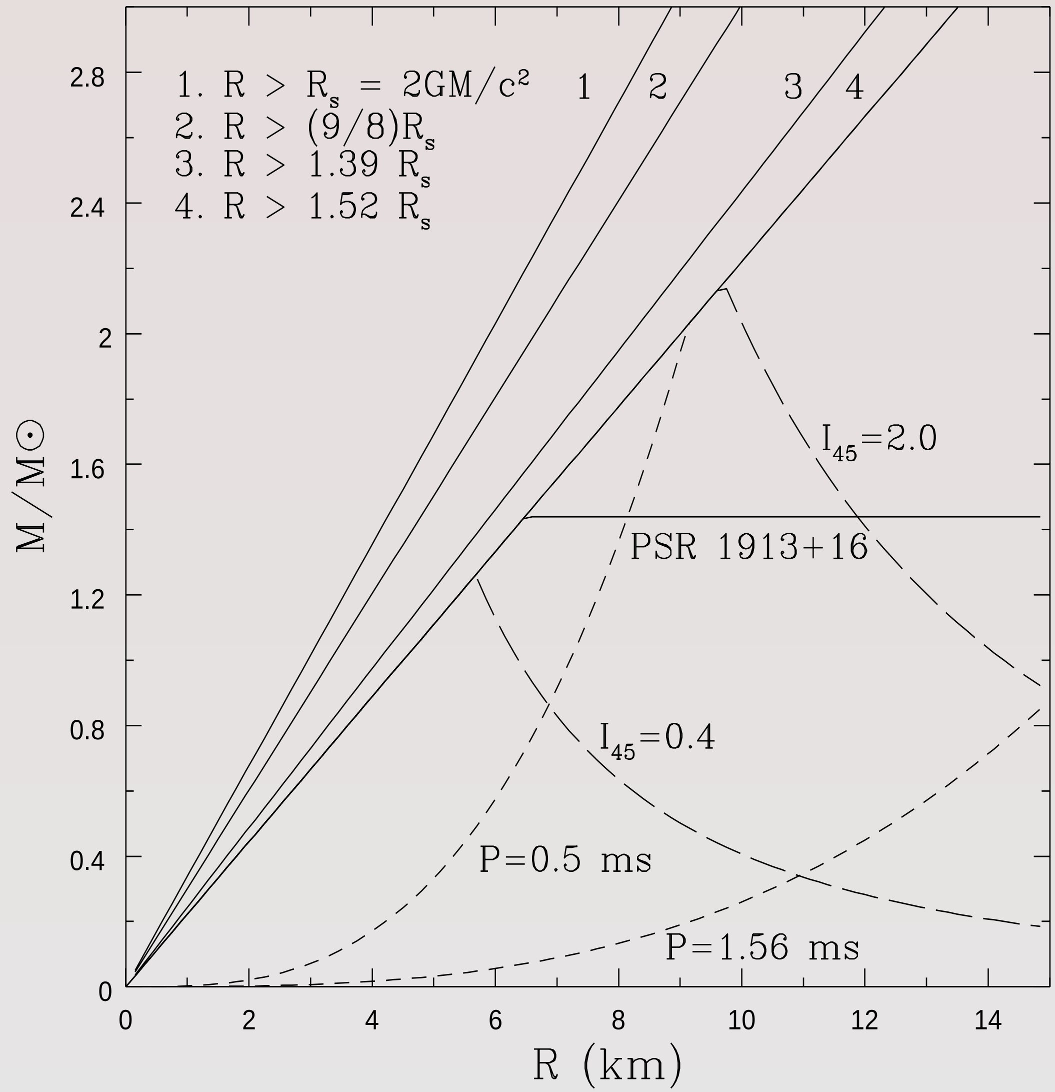
- Energy (E) & Pressure (P) vs. scaled density ($u = n/n_0$).
- Nuclear matter equilibrium density $n_0 = 0.16 \text{ fm}^{-3}$.
- Proton fraction $x = n_p/(n_p + n_n)$.
- Nuclear matter : $x = 1/2$.
- Neutron matter : $x = 0$.
- Stellar matter in $\beta-$ equilibrium : $x = \tilde{x}$.

Results of Star Structure



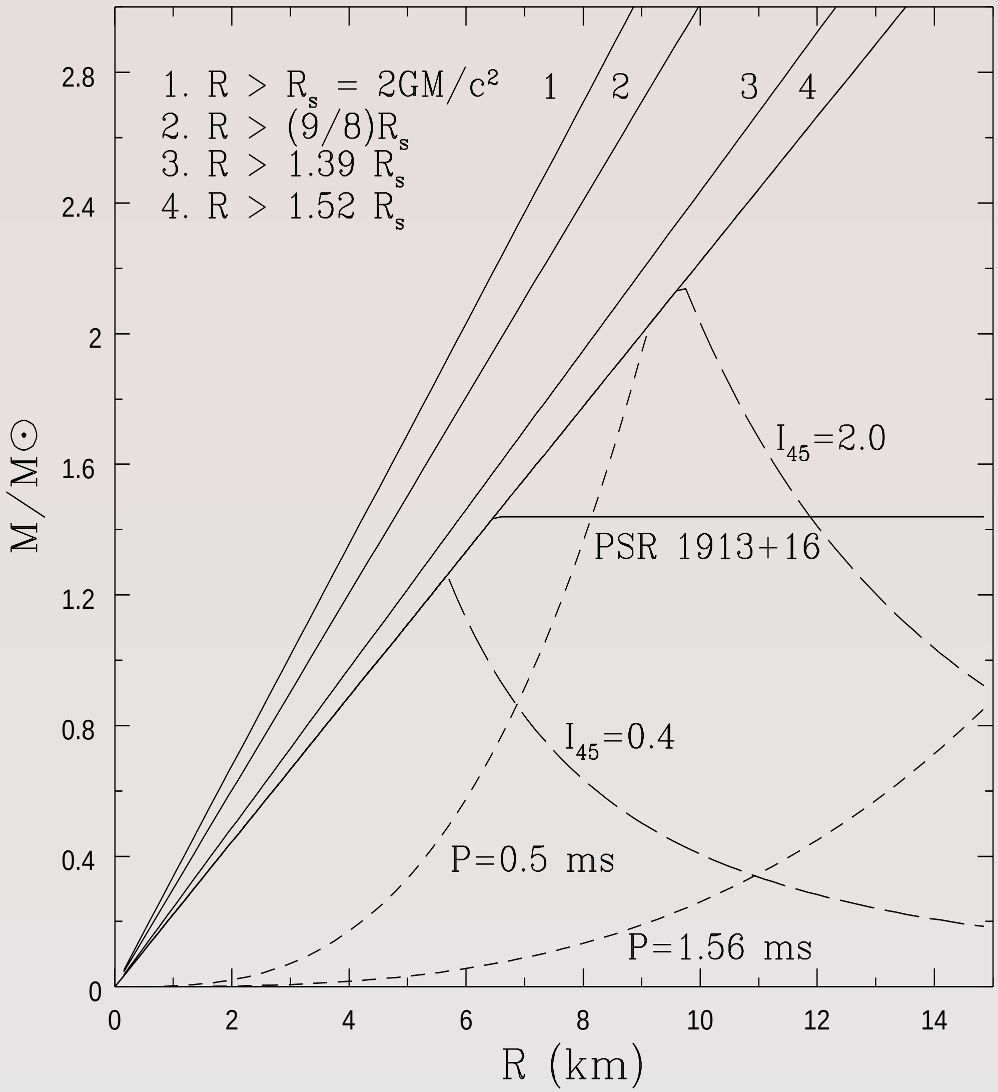
- Stellar properties for soft & stiff (by comparison) EOS's.
- Causal limit : $P = \epsilon$.
- M_g : Gravitational mass
- R : Radius
- BE : Binding energy
- n_b : Central density
- I : Moment of inertia
- ϕ : Surface red shift , $e^{\phi/c^2} = (1 - 2GM/c^2R)^{-1/2}$.

Constraints on the EOS-I



- $R > R_s = 2GM/c^2 \Rightarrow$
  $M/M_\odot \geq R/R_{s\odot}$;
  $R_{s\odot} = 2GM_\odot/c^2$
  $\simeq$ 2.95 km .
- $P_c < \infty$
  $\Rightarrow R > (9/8)R_s$
  $\Rightarrow M/M_\odot \geq$
  $(8/9)R/R_{s\odot}$.
- Sound speed c_s :
  $c_s = (dP/d\epsilon)^{1/2} \leq c$
  $\Rightarrow R > 1.39R_s$
  $\Rightarrow M/M_\odot \geq$
  $R/(1.39R_{S\odot})$.
- If $P = \epsilon$ above
  $n_t \simeq 2n_0$,
  $R > 1.52R_s \Rightarrow$
  $M/M_\odot R/(1.52R_{s\odot})$.

Constraints on the EOS-II



- $M_{max} \geq M_{obs}$;
  In PSR 1913+16,
  $M_{obs} = 1.44\ \mathrm{M}_\odot$.
- In PSR 1957+20,
  $P_K = 1.56$ ms :
  $\Omega_K \simeq 7.7 \times 10^3$
  $\left(\frac{M_{max}}{\mathrm{M}_\odot}\right)^{1/2} \left(\frac{R_{max}}{10\ \mathrm{km}}\right)^{-3/2}\ \mathrm{s}^{-1}$
- Mom. of Inertia I :
  $I_{max} = 0.6{\times}10^{45}\ \mathrm{g\ cm}^2$
  $\left(\frac{M_{max}}{\mathrm{M}_\odot}\right) \left(\frac{R_{max}}{10\ \mathrm{km}}\right)^2$
  $f(M_{max}, R_{max})$
- In SN 1987A
  $B.E. \simeq (1-2)$
  $\times 10^{53}$ ergs.

Composition of Dense Stellar Matter

- Crustal Surface :
  electrons, nuclei, dripped neutrons, $\cdots$ set in a lattice
  new phases with lasagna, sphagetti, $\cdots$ like structures
- Liquid (Solid?) Core :
  $n, p, \Delta, \cdots$ leptons: $e^{\pm}, \mu^{\pm}, \nu_e' s, \nu_\mu' s$
  $\Lambda, \Sigma, \Xi, \cdots$
  $K^-, \pi^-, \cdots$ condensates
  $u, d, s, \cdots$ quarks
- Constraints :
  1. $n_b = n_n + n_p + n_\Lambda + \cdots$: baryon # conservation
  2. $n_p + n_{\Sigma^+} + \cdots = n_e + n_\mu$: charge neutrality
  3. $\mu_i = b_i \mu_n - q_i \mu_\ell$: energy conservation

$\Rightarrow$

$$\mu_\Lambda = \mu_{\Sigma^0} = \mu_{\Xi^0} = \mu_n \quad \mu_{\Sigma^-} = \mu_{\Xi^-} = \mu_n + \mu_e \quad \mu_p = \mu_{\Sigma^+} = \mu_n - \mu_e$$

$\Rightarrow$

$$\mu_{K^-} = \mu_e = \mu_\mu = \mu_n - \mu_p$$

$\Rightarrow$

$$\mu_d = \mu_u + \mu_e = \mu_s = (\mu_n + \mu_e)/3$$
$$\mu_u = (\mu_n - 2\mu_e)/3$$

Nuclear Matter-I

Consider equal numbers neutrons (N) and protons (Z) in a large volume V at zero temperature ($T = 0$).

Let $n = (N + Z)/V = n_n + n_p$ denote the neutron plus proton number densities; $n = 2k_F^3/(3\pi^2)$, where k_F is the Fermi momentum.

Given the energy density $\epsilon(n)$ inclusive of the rest mass density mn, denote the energy per particle by $E/A = \epsilon/n$, where $A = N + Z$.

Pressure: From thermodynamics, we have

$$\begin{aligned} P = & \quad -\frac{\partial E}{\partial V} \quad = \; - \; \frac{dE}{d(A/n)} \\ = & \quad n^2 \frac{d(\epsilon/n)}{dn} \quad = \; n \, \frac{d\epsilon}{dn} - \epsilon = n\mu - \epsilon \, , \end{aligned}$$

where $\mu = d\epsilon/dn$ is the chemical potential inclusive of the rest mass m. At the equilibrium density n_0, where $P(n_0) = 0$, $\mu = \epsilon/n = E/A$.

Nuclear Matter-II

Incompressibility: The compressibility χ is usually defined by

$$\chi = -\frac{1}{V}\frac{\partial V}{\partial P} = \frac{1}{n}\left(\frac{dP}{dn}\right)^{-1}$$

However, in nuclear physics applications, the incompressibility factor

$$K(n) = 9\,\frac{dP}{dn} = 9\,n\frac{d^2\epsilon}{dn^2}\,, \qquad \text{or}$$

$$= 9\,\frac{d}{dn}\left[n^2\frac{d(E/A)}{dn}\right] = 9\,\left[n^2\frac{d^2(E/A)}{dn^2} + 2n\frac{d(E/A)}{dn}\right]$$

is used. At the equilibrium density n_0, the compression modulus

$$K(n_0) = 9n_0^2\,\left.\frac{d^2(E/A)}{dn^2}\right|_{n_0} = k_F^{0^2}\,\left.\frac{d^2(E/A)}{dk_F^2}\right|_{k_F^0}\,.$$

Above, $k_F^0 = (3\pi^2 n_0/2)^{1/3}$ denotes the equilibrium Fermi momentum.

Nuclear Matter-III

Adiabatic sound speed: The propagation of small scale density fluctuations occurs at the sound speed obtained from the relation

$$\left(\frac{c_s}{c}\right)^2 = \frac{dP}{d\epsilon} = \frac{dP/dn}{d\epsilon/dn} = \frac{1}{\mu}\frac{dP}{dn} = \frac{d\ln\mu}{d\ln n}\,.$$

Alternative relations for the sound speed squared are

$$\left(\frac{c_s}{c}\right)^2 = \frac{K}{9\mu} = \Gamma\frac{P}{P+\epsilon}\,,$$

where $\Gamma = d\ln P/d\ln\epsilon$ is the adiabatic index. It is desirable to require that the sound speed does not exceed that of light.

Schematic Nuclear Matter EOS-I

$$\epsilon(n) = n\left[m + \frac{3}{5}\frac{\hbar^2 k_F^2}{2m} + \frac{A}{2}u + \frac{B}{\sigma+1}u^\sigma\right],$$

m := nucleon mass ,

$u = n/n_0$:= density compression ratio ,

Second term := mean kinetic energy of isospin symmetric matter ,

Last two terms := parametrize contributions from density dependent potential interactions. Such terms arise, for example, when local contact interactions are used to model the nuclear forces.

The coefficients A, B and σ can be determined by using the empirically determined properties of nuclear matter at the equilibrium density.

Schematic Nuclear Matter EOS-II

Using the relations for the state variables, and recalling that the total baryon density $n = 2k_F^3/3\pi^2$, we get

$$\begin{aligned}
\frac{E}{A} - m = \quad & \frac{\epsilon}{n} - m & = \langle E_F^0 \rangle u^{2/3} + \frac{A}{2} u + \frac{B}{\sigma + 1} u^\sigma \\
\frac{P}{n_0} = \quad & u^2 \frac{d(E/A)}{du} & = \frac{2}{3} \langle E_F^0 \rangle u^{5/3} + \frac{A}{2} u^2 + \frac{B\sigma}{\sigma + 1} u^{\sigma+1} \\
\frac{K}{9} = \quad & \frac{d(P/n_0)}{du} & = \frac{10}{9} \langle E_F^0 \rangle u^{2/3} + Au + B\sigma\, u^\sigma \,.
\end{aligned}$$

Above, $\langle E_F^0 \rangle = (3/5)(\hbar k_F^0)^2/2m$ is the mean kinetic energy per particle

at equilibrium, with k_F^0 denoting the Fermi momentum at equilibrium.

Schematic Nuclear Matter EOS-III

Setting $u = n/n_0 = 1$ for which $P/n_0 = 0$, we find

$$\sigma = \frac{K_0 + 2\langle E_F^0 \rangle}{9\left[\frac{1}{3}\langle E_F^0 \rangle - \left(\frac{E}{A} - m\right)\right]}$$

$$B = \left(\frac{\sigma + 1}{\sigma - 1}\right)\left[\frac{1}{3}\langle E_F^0 \rangle - \left(\frac{E}{A} - m\right)\right]$$

$$A = \left[\left(\frac{E}{A} - m\right) - \frac{5}{3}\langle E_F^0 \rangle\right] - B$$

Choosing $E/A - m = -16$ MeV and $n_0 = 0.16\ \mathrm{fm}^{-3}$ leads to:

K_0 (MeV)	A (MeV)	B (MeV)	σ
200	-366.23	313.39	1.16
400	-122.17	65.39	2.11

Limitations to watch out for

- Since $\sigma > 1$ for both choices of K_0, the energy density varies more rapidly than n^2 which leads to an acausal behavior in the EOS.

- Also, input values of K_0 below a certain limit are not allowed.

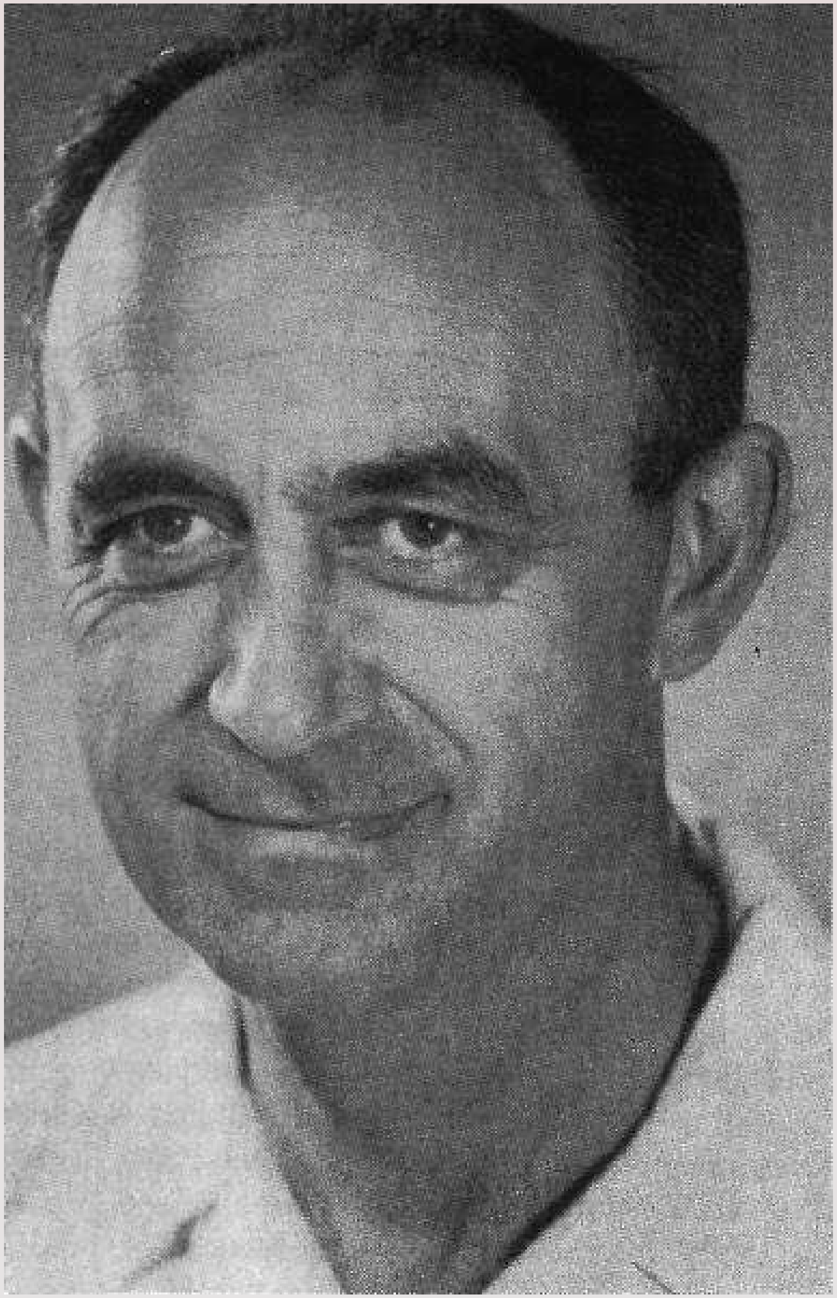
An adequate parametrization that reproduces the nuclear matter energy density of realistic microscopic calculations is provided by

$$\epsilon_{nm} = mn_0u + \frac{3}{5}E_F^0 n_0 u^{5/3} + V(u)\,,$$

where the potential contribution $V(u)$ is given by

$$\begin{aligned} V(u) &= \frac{1}{2}An_0u^2 + \frac{Bn_0u^{\sigma+1}}{1+B'u^{\sigma-1}} \\ &+ u\sum_{i=1,2} C_i 4\int \frac{d^3k}{(2\pi)^3}\, g(k,\Lambda_i)\theta(k_F-k)\,. \end{aligned}$$

The function $g(k,\Lambda_i)$ is suitably chosen to simulate finite range effects.



Neutron-rich Matter-I

- $\alpha = (n_n - n_p)/n$:= excess neutron fraction
- $n = n_n + n_p$:= total baryon density
- $x = n_p/n = (1-\alpha)/2$:= proton fraction

The neutron and proton densities are then

$$n_n = \frac{(1+\alpha)}{2}\, n = (1-x)\, n \quad \& \quad n_p = \frac{(1-\alpha)}{2}\, n = x\, n\,.$$

For nuclear matter, $\alpha = 0$ $(x = 0.5)$, whereas,
for pure neutron matter, $\alpha = 1$ $(x = 0)$.

Write the energy per particle (by simplifying E/A to E) as

$$E(n,\alpha) = E(n,\alpha=0) + \ \Delta E_{kin}(n,\alpha) + \Delta E_{pot}(n,\alpha)\,, \quad \text{or}$$

- 1st term := energy of symmetric nuclear matter
- 2nd & 3rd terms := isospin asymmetric parts of kinetic and interaction terms in the many–body hamiltonian

Neutron-rich Matter-II

In a non–relativistic description,

$$\begin{aligned}\epsilon_{kin}(n,\alpha) &= \frac{3}{5}\frac{\hbar^2}{2m}\left[(3\pi^2 n_n)^{2/3}\, n_n + (3\pi^2 n_p)^{2/3}\, n_p\right] \\ &= n\,\langle E_F\rangle \cdot \frac{1}{2}\left[(1+\alpha)^{5/3} + (1-\alpha)^{5/3}\right] .\end{aligned}$$

- $\langle E_F\rangle = (3/5)(\hbar^2/2m)(3\pi^2 n/2)^{2/3}$:= mean K.E. of nuclear matter.

$$\begin{aligned}\Delta E_{kin}(n,\alpha) &= E_{kin}(n,\alpha) - E_{kin}(n,\alpha=0) \\ &= = \frac{1}{3}\, E_F \cdot \alpha^2 \left(1 + \frac{\alpha^2}{27} + \cdots\right) .\end{aligned}$$

- Quadratic term above offers a useful approximation ;
- From experiments, bulk symmetry energy $\simeq 30$ MeV ;
- Contribution from K.E. amounts to $E_F^0/3 \simeq (12-13)$ MeV ;
- Interactions contribute more to the total bulk symmetry energy .

Neutron-rich Matter-III

$$E(n,x) = E(n,1/2) + S_2(n)\,(1-2x)^2 + S_4(n)\,(1-2x)^4 + \cdots \,.$$

- $S_2(n), S_4(n), \cdots$ from microscopic calculations.

Chemical Potentials :

Utilizing $E = \epsilon/n$, $n = n_n + n_p$, $x = n_p/n$, and $u = n/n_0$,

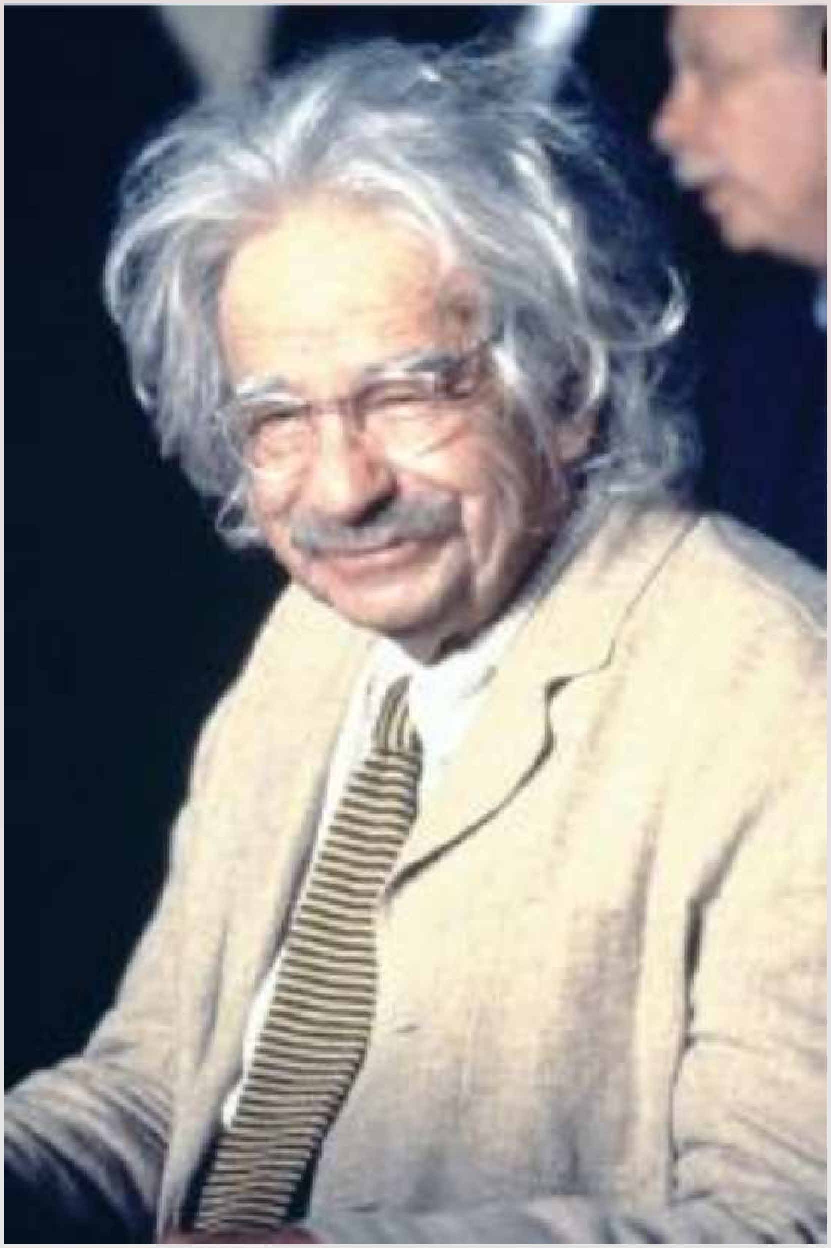
$$\mu_n = \left.\frac{\partial \epsilon}{\partial n_n}\right|_{n_p} = E + u\left.\frac{\partial E}{\partial u}\right|_x - x\left.\frac{\partial E}{\partial x}\right|_n \,,$$

$$\mu_p = \left.\frac{\partial \epsilon}{\partial n_p}\right|_{n_n} = \mu_n + \left.\frac{\partial E}{\partial x}\right|_n \,,$$

$$\widehat{\mu} = \mu_n - \mu_p = -\left.\frac{\partial E}{\partial x}\right|_n$$

$$= 4(1-2x)\,\left[S_2(n) + 2S_4(n)\,(1-2x)^2 + \cdots\right] \,.$$

- $\widehat{\mu}$ determines the composition of charge neutral neutron star matter.
- $\widehat{\mu}$ governed by the density dependence of the symmmetry energy.



Charge neutral neutron-rich matter-I

- Old neutron stars are in equilibrium w.r.t. weak interactions.

- The ground state consists of strongly interacting hadrons and weakly interacting leptons generated in β-decays and inverse β-decays.

- Local charge neutrality and chemical equilibrium prevail.

First consider matter with neutrons, protons and electrons involved in

$$\begin{aligned} n\,(\text{or}\;+n) &\rightarrow p\,(\text{or}\;+n) + e^- + \overline{\nu}_e\,, \\ p\,(\text{or}\;+n) + e^- &\rightarrow n\,(\text{or}\;+n) + \nu_e \end{aligned}$$

- In cold catalyzed neutron star matter, neutrinos leave the system leading to

$$\widehat{\mu} = \mu_n - \mu_p = \mu_e\,.$$

(Later we will also consider the case in which neutrinos are trapped in matter, which leads instead to $\mu_n - \mu_p = \mu_e - \mu_{\nu_e}$.)

Charge neutral neutron-rich matter-II

- In beta equilibrium, one has

$$\frac{\partial}{\partial x}\left[E_b(n,x)+E_e(x)\right]=0\,.$$

- Charge neutrality implies that $n_e = n_p = nx$, or, $k_{F_e} = k_{F_p}$.

Combining these results, $\widetilde{x}(n)$ is determined from

$$4(1-2x)\;\left[S_2(n)+2S_4(n)\,(1-2x)^2+\cdots\right]=\hbar c\left(3\pi^2 nx\right)^{1/3}\,.$$

When $S_4(n) << S_2(n)$, $\widetilde{x}$ is obtained from $\beta\,\widetilde{x}-(1-2\widetilde{x})^3=0\,,$
where $\beta = 3\pi^2 n\,(\hbar c/4S_2)^3$. Analytic solution ugly!

For $u \leq 1$, $\widetilde{x} << 1$, and to a good approximation $\widetilde{x} \simeq (\beta+6)^{-1}\,.$

- Notice the high sensitivity to $S_2(n)$, which favors the addition of protons to matter.

Charge neutral neutron-rich matter-III

Muons in matter :

When $E_{F_e} \geq m_\mu c^2 \sim 105$ MeV, electrons to convert to muons through

$$e^- \longrightarrow \mu^- + \overline{\nu}_\mu + \nu_e \,.$$

Chemical equilibrium implies $\mu_\mu = \mu_e$.
At threshold, $\mu_\mu = m_\mu c^2 \sim 105$ MeV.

As the proton fraction at nuclear density is small, $4S_2(u)/m_\mu c^2 \sim 1$.
Using $S_2(u=1) \simeq 30$ MeV, threshold density is $\sim n_0 = 0.16 \text{ fm}^{-3}$.
Above threshold,

$$\mu_\mu = \sqrt{k_{F_\mu}^2 + m_\mu^2 c^4} = \sqrt{(\hbar c)^2 (3\pi^2 n x_\mu)^{2/3} + m_\mu^2 c^4} \,.$$

- $x_\mu = n_\mu / n_b$:= muon fraction in matter.

The new charge neutrality condition is $n_e + n_\mu = n_p$.
Muons make $x_e = n_e/n_b$ to be lower than its value without muons.

Charge neutral neutron-rich matter-IV

Total energy density & pressure :

$$\epsilon_{tot} = \epsilon_b + \sum_{\ell=e^-,\mu^-} \epsilon_\ell \quad \& \quad P_{tot} = P_b + \sum_{\ell=e^-,\mu^-} P_\ell$$

- $\epsilon_{b,\ell}$ and $P_{b,\ell}$:= energy density and pressure of baryons (leptons).

$$\epsilon_\ell = 2\int \frac{d^3k}{(2\pi)^3}\sqrt{k^2+m_\ell^2} \quad \& \quad P_\ell = \frac{1}{3}\cdot 2\int \frac{d^3k}{(2\pi)^3}\frac{k^2}{\sqrt{k^2+m_\ell^2}}$$

$$\begin{aligned}
\epsilon_b &= mn_0u + \left\{\frac{3}{5}E_F^0 n_0 u^{5/3} + V(u)\right\} + n_0(1-2x)^2 uS(u)\,, \\
P_b &= \left\{\frac{2}{5}E_F^0 n_0 u^{5/3} + \left(u\frac{dV}{du} - V\right)\right\} + n_0(1-2x)^2 u^2 \frac{dS}{du}\,.
\end{aligned}$$

- As $\alpha_{em} \simeq 1/137$, free gas expressions for leptons are satisfactory.

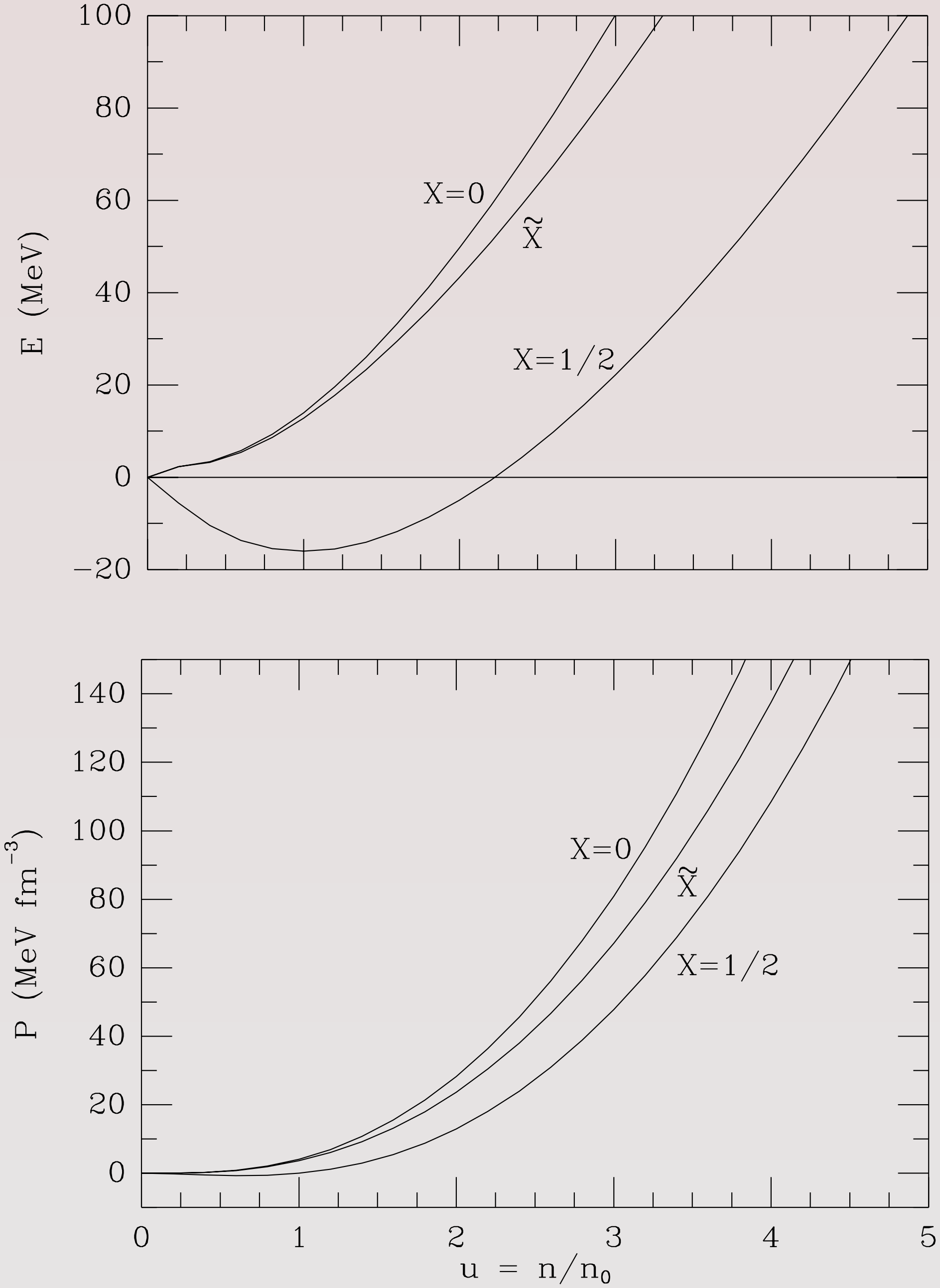
Charge neutral neutron-rich matter-V

STATE VARIABLES AT NUCLEAR DENSITY

Quantity	Nuclear matter	Stellar matter
$\tilde{x}$	0.5	0.037
$\epsilon_b/n - m$	-16	9.6
ϵ_e/n	0	3.18
P_b	0	3.5
P_e	0	0.17
$\mu_n - m$	-16	35.74
$\mu_p - m$	-16	-75.14
$\mu_e = \mu_n - \mu_p$	0	110.88

Energies in MeV and pressure in MeV fm^{-3}. The numerical estimates are based on an assumed symmetry energy $S_2(u) = 13u^{2/3} + 17u$, where $u = n/n_b$.

Nucleonic Equation of State



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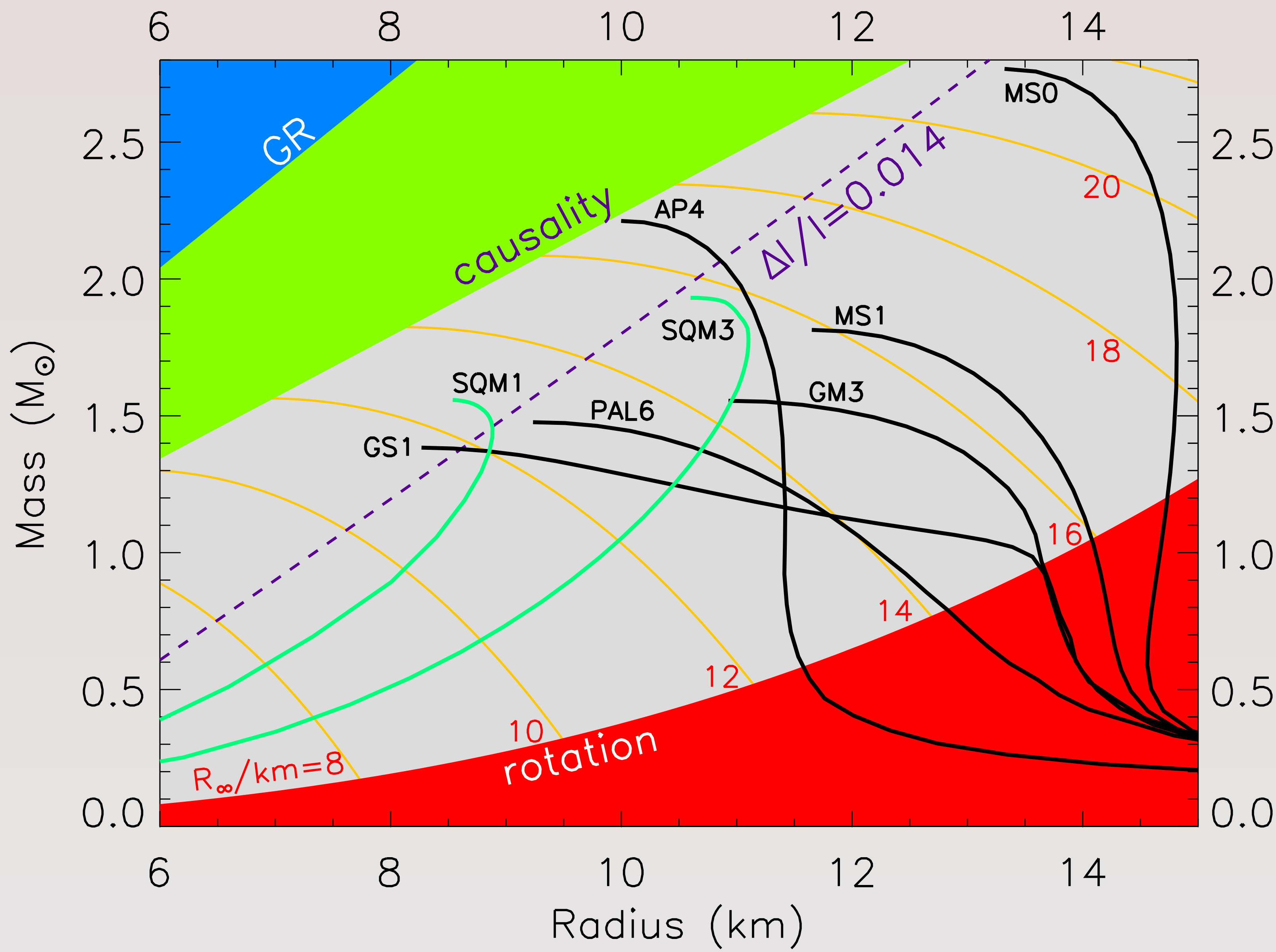
Mass Radius Relationship



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