Lectures on effective field theory

David B. Kaplan

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Abstract

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Contents

1 Introduction to effective field theory

1.1 What is an effective field theory?

The content of a quantum theory can be encoded in correlation functions which are in general very complicated functions of the momenta of incoming and outgoing particles. In particular they exhibit cuts and poles and various other nonanalytic behavior, which arise when the kinematics allow for physical intermediate states. When that happens, the $1/(p^2 - m^2)$ propagators for the intermediate states become singular which leads to the nonanalytic behavior of the correlation function [1]. Once you realize that this is the source of most of the complicated behavior in a correlation function, it is apparent that when the kinematics are far from being able to produce a propagating heavy state, the contribution of that heavy state to the correlation function of interest will be relatively simple, well approximated the first few terms in a Taylor expansion in the incoming momenta of the scattering problem. For example, when considering neutron decay, in which (at tree level in the Standard Model) a d quark decays into a u quark and a virtual W, which then turns into an electron and anti-neutrino. The energy released is about 10^{-6} times the W mass, and so the W propagator may be approximated by $1/(p^2 - M_W^2) \simeq -1/M_W^2 - p^2/M_W^4 + \ldots$. While such a Taylor expansion makes the process slightly simpler to analyze, the benefits seem minimal, and it does not seem obvious how to generalize the procedure to loop diagrams, or nonperturbative physics.

Instead of Taylor expanding each amplitude, to simplify the contributions from heavy states to a low energy process, it turns out to be much more profitable instead to construct a low energy Lagrangian from which all low energy processes may be computed while accounting for heavy physics in a Taylor expansion in external momenta divided by the heavy mass scale. Such a Lagrangian is called an effective field theory, but it is just as useful in nonrelativistic physics as in quantum field theory. An effective field theory consists in principle of an infinite sum of local operators which are typically of diminishing importance with higher dimension. In practice this series is truncated according to the accuracy desired for the process in question.

There are many situations in which effective field theories are of interest:

- They allow one better understand complicated problems involving a lot of different length scales, and to understand the results qualitatively from dimensional analysis;
- They allow one to compute low energy scattering amplitudes without having a detailed understanding of short distance physics, or to avoid wasting time calculating tiny effects from known short distance physics;
- In nonperturbative theories (such as low energy QCD) one can construct a predictive effective field theory for low energy phenomena by combining a power counting of operators with the symmetry constraints of the underlying theory (such as the chiral Lagrangian for pion physics).

[1] This discussion is for a relativistic theory, but analogous statements may be made for nonrelativistic theories.

- By regarding theories of known physics as effective field theory descriptions of more fundamental underlying physics, one can work from bottom up, extrapolating from observed rare processes to a more complete theory of short distance physics.

In these lectures I will try to explain in more detail what is an effective theory, how to construct it, and how to use it. This is a very big subject on which there are a number of good reviews. Each is idiosyncratic, and I recommend that you read a number of them to gain perspective and encounter a wider range of applications. Favorites mine are found in refs. ??

1.2 Local operators and scaling dimension in a relativistic theory

Let's start off by just discussing nucleon-nucleon scattering. If an effective field theory is given by a Lagrangian with an infinite number of operators in it, to be useful there must be a criterion for ignoring most of them. So as a prototypical example of such a theory, consider the Lagrangian (in four dimensional Euclidean spacetime, after a Wick rotation to imaginary time) for relativistic scalar field with a $\phi \to -\phi$ symmetry:

$$\mathcal{L}_E = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \sum_n \left(\frac{c_n}{\Lambda^{2n}}\phi^{4+2n} + \frac{d_n}{\Lambda^{2n}}(\partial\phi)^2\phi^{2+2n} + \ldots \right) \tag{1}$$

I have introduced a scale Λ, the momentum cutoff of the theory in a particular way to make the the couplings l, c_n and d_n all dimensionless, since the mass dimension of ϕ is 1, while the mass dimension of $\mathcal{L}$ is 4. [2]. I will assume that $\lambda \ll 1$, $c_n \ll 1$ and $d_n \ll 1$.

So how do we figure out which of the interactions in $\mathcal{L}$ are the most important? To compute correlation functions in this theory in d dimensions, one performs the path integral:

$$\int D\phi\, e^{-S_E} \ , \qquad S_E = \int d^dx\, \mathcal{L}_E \ . \tag{2}$$

Now consider a particular field configuration $\tilde\phi$ that contributes to this path integral, where $\tilde\phi$ is localized to a spacetime volume of size L^4, where $L \simeq 2\pi/k$ with wavenumber $|k_\mu| \sim k$, and has amplitude ϕ_k. See figure 1, where I plotted a "wavelet". With this configuration, the Euclidean action is given by

$$S_E \simeq (2\pi)^4 \left[\frac{\widehat{\phi}_k^2}{2} + \frac{m^2}{k^2}\widehat{\phi}_k^2 + \frac{\lambda}{4!}\widehat{\phi}_k^4 + \sum_n \left(c_n \left(\frac{k^2}{\Lambda^2}\right)^n \widehat{\phi}_k^{4+2n} + d_n \left(\frac{k^2}{\Lambda^2}\right)^n \widehat{\phi}_k^{4+2n} + \ldots \right) \right] , \tag{3}$$

where

$$\widehat{\phi}_k \equiv \phi_k/k \ . \tag{4}$$

[2] To determine the mass dimension, start with the fact that relativity requires p and E to have mass dimension one, so that the uncertainty relations with $\hbar = 1$ require x and t to have mass dimension -1. Then since the action $S = \int d^dx\, \mathcal{L}$ in d-dimensions, must be dimensionless, it follows that $(\partial\phi)^2$ is dimensionless in d dimensions, and so a scalar field has mass dimension $(d/2 - 1)$.

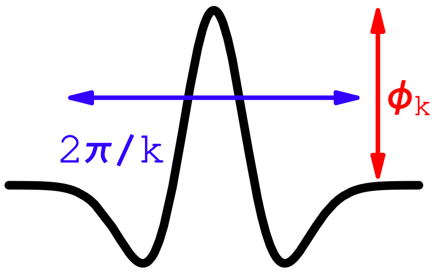

Figure 1: *A sample configuration contributing to the path integral for the scalar field theory in eq. (1). Its amplitude is ϕ_k and has wave number $\sim k$ and spatial extent $\sim 2\pi/k$.*

Now for the path integral, consider the ordinary integration over the amplitude $\widehat{\phi}_k$ for a particular k:

$$\int d\widehat{\phi}_k \, e^{-S_E} \ . \tag{5}$$

The integral is dominated by those values of $\widehat{\phi}_k$ for which $S_E \lesssim 1$. Which are the important terms in S_E in this region? First, assume that the particle is relativistic, $k \gg m$. Then evidently, as the amplitude $\widehat{\phi}_k$ gets large, the first term in S_E to become become large is the kinetic term, $(2\pi)^4\widehat{\phi}_k^2$. It determines that the integral gets its maximum contribution at $\phi_k \sim k/(2\pi)^2$. It is because the kinetic term controls the fluctuations of the scalar field that we "canonically normalize" the field such that the kinetic term is $\frac{1}{2}(\partial\phi)^2$, and perturb in the coefficients of the other operators in the theory.

Since we assume that $k \leq \Lambda$, Λ being the momentum cutoff, and that the c_n and d_n couplings are $<< 1$, we see that the integrand will become small when the terms in S_E which are quadratic in $\widehat{\phi}_k$ become $O(1)$, at which point the terms with higher powers of $\widehat{\phi}_k$ are still small.

What happens as we vary k? We see from eq. (3) that as k is reduced, the c_n and d_n terms, proportional to $(k^2/\Lambda^2)^n$, get smaller. Such operators are termed "irrelevant" in Wilson's language, because they become unimportant in the infrared (low k). In contrast, the mass term becomes more important; it is called a "relevant" operator. The kinetic term and the $\lambda\phi^4$ interaction do not change; such operators are called "marginal".

Classification of operators as irrelevant, marginal, or relevant may be done in a more straightforward way by a simple scaling exercise. Consider an arbitrary field configuration $\phi(x)$ contributing to the path integral. The action for this field configuration is $S_E(\phi(x); m, \lambda, c_n, d_n, \ldots)$, where the functional S_E is given in eq. (3). I have made a point of listing as its arguments not only the field ϕ, but all of the couplings that characterize the theory. Now consider the family of field configurations

$$\phi_\xi(x) = \phi(\xi x) \ . \tag{6}$$

For example, if $\phi(x) = e^{ik\cdot x}$, then $\phi_\xi(x) = e^{i\xi k\cdot x}$, so that $\xi \to 0$ corresponds to looking at configurations with longer wavelength $k' = \xi k$ (the "IR"). The action for this family of configurations is

$$
\begin{aligned}
S_E(\phi_\xi(x);\Lambda,m^2,&\lambda,c_n,d_n,\ldots)\\
&= \int d^4x\, \frac{1}{2}(\partial_x\phi(\xi x))^2 + \frac{1}{2}m^2\phi(\xi x)^2 + \frac{\lambda}{4!}\phi(\xi x)^4\\
&\quad + \sum_n c_n \frac{\phi^{4+2n}(\xi x)}{\Lambda^{2n}} + d_n\frac{(\partial_x\phi(\xi x))^2\phi^{2n}(\xi x)}{\Lambda^{2n}}\\
&= \int d^4x'\, \frac{1}{2}(\partial_{x'}\phi'(x'))^2 + \frac{1}{2}m^2\xi^{-2}\phi'(x')^2 + \frac{\lambda}{4!}\phi'(x')^4\\
&\quad + \sum_n c_n\xi^{2n} \frac{\phi'^{4+2n}(x')}{\Lambda^{2n}} + d_n\xi^{2n}\frac{(\partial_{x'}\phi'(x'))^2\phi'^{2n}(x')}{\Lambda^{2n}}\ ,
\end{aligned}
\tag{7}
$$

where $\phi'(x) \equiv \xi^{-1}\phi(x)$, and I made the change of integration variable $x' = \xi x$. Since x' is a dummy variable, we can drop the prime and recognize that the above action equals the original action with rescaled fields and couplings:

$$
S_E(\phi(\xi x);\Lambda,m^2,\quad \lambda,c_n,d_n,\ldots) = S_E\left(\xi^{-1}\phi(x);\xi^{-2}m^2,\lambda,c_n\xi^{2n},d_n\xi^{2n},\ldots\right)\ . \tag{8}
$$

so that

$$
\phi\to\xi^{-1}\phi\ ,\quad m^2\to\xi^{-2}m^2\ ,\quad \lambda\to\lambda\ ,\quad c_n\to\xi^{2n}c_n\ ,\quad d_n\to\xi^{2n}d_n\ . \tag{9}
$$

Now as we scale to the infrared (long wavelength, low energy processes) by taking $\xi\to 0$, we see again that the mass term grows in importance (relevant), the c_n and d_n couplings fall like ξ^{2n} (irrelevant, with higher n being more irrelevant), and the kinetic terms and $\lambda\phi^4$ interaction not changing (marginal). Using this analysis one can also easily see that if there was a constant $\mathcal{E}_0$ added to our Lagrangian (vacuum energy, or a cosmological constant term), it would scale as $\mathcal{E}_0\to\xi^{-4}\mathcal{E}_0$, and hence would be very relevant.

It is convenient to define a scaling dimension, which is the negative of the power of ξ with which a quantity scales, and I will denote the this scaling dimension with square brackets $[\cdots]$. Thus $[x] = -1$ and eq. (9) tells us

$$
[\phi] = 1\ ,\quad [m^2] = 2\ ,\quad [\lambda] = 0\ ,\quad [c_n] = [d_n] = -2n\ . \tag{10}
$$

Marginal operators have coefficients with scaling dimension zero; coefficients of relevant operators have positive scaling dimensions; irrelevant operators have negative scaling dimension. Note that this scaling dimension for a classical, relativistic action is just the same as the mass dimension when $\hbar = c = 1$.

You might wonder about the above derivation: why did I choose to scale $\phi' = \xi^{-1}\phi$ instead of some other power? Then the kinetic term would have picked up powers of ξ, but other operators might not ave, depending on the choice of scaling for ϕ. The answer is that the kinetic term was chosen to be the scale invariant term because, for a weakly interacting relativistic system, it is the kinetic term that dominates the size of fluctuations in the path integral (see eq. (3)). This sort of scaling argument will clearly fail when $k \lesssim m$, at which point the mass term dominates. So we will have to provide a different scaling argument for nonrelativistic theories.

1.2.1 Scaling in a nonrelativistic theory

You might complain at this point that what I termed marginal depended crucially on the fact that the field ϕ_ξ involved rescaling ϕ by s; if I had chosen a different power, I could have ended up with a different classification of operators. That choice of power though was made in order to keep the kinetic term unchanged. This was warranted by the fact that as we saw above, the kinetic term is what controls the size of fluctuations of ϕ, so long as $k \gtrsim m$. However we can expect a different scaling law to hold for nonrelativistic particles, when the mass term dominates.

Going back to our wavelets, what happens happens when k is taken below m, and the particle becomes nonrelativistic? In Minkowski spacetime we have $\mathcal{L} = \frac{1}{2}(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2 - \ldots)$, and for a nonrelativistic particle ϕ has the expansion $\phi = (a_k\, e^{-iEt} + a_k^\dagger e^{iEt})$, with $E \simeq m$. Note that since $E \simeq m$, both $\dot{\phi}^2$ and $m^2\phi^2$ are large but nearly cancel in the Minkowski action. This confuses the power counting. Therefore it is convenient to change variables to absorb the uninteresting rapid oscillation associated with the particle's rest energy. This leads us to the substitution

$$\phi(x,t) = \frac{1}{\sqrt{2m}}\left(e^{-imt}\psi(x,t) + c.c\right) \ , \tag{11}$$

where ψ is complex and $\dot{\psi} \ll m\psi$. Because of this last condition, we can drop from our Lagrangian terms with unequal numbers of ψ and ψ^*, since they would vanish when the Lagrangian is integrated over time. The Minkowski spacetime Lagrangian may now be rewritten as

$$\mathcal{L}_M = \psi^*\left(i\partial_t + \frac{\nabla^2}{2m}\right)\psi - \frac{\lambda}{8m^2}(\psi^\dagger\psi)^2 + \ldots \tag{12}$$

and the Euclidean action is given by

$$S_E(\psi(x,t);\lambda,\ldots) = \int dt\, d^3x\, \psi^*\left(\partial_t - \frac{\nabla^2}{2m}\right)\psi + \frac{\lambda}{8m^2}(\psi^\dagger\psi)^2 + \ldots \tag{13}$$

The ∂_t and $\frac{\nabla^2}{2m}$ operators must be considered to be of the same size, one corresponds to the energy E and the other to $p^2/2m$. Thus to analyze how things scale, we must have time and space scale differently, namely $x \to x' = \xi x$, $t \to t' = \xi^2 t$. As before, we determine the scaling properties of ψ by requiring that the kinetic term be scale invariant, because we believe that it dominates the fluctuations in the path integral. Thus we consider the family of field configurations

$$\psi_\xi(x,t) = \psi(\xi x, \xi^2 t) \ . \tag{14}$$

The new action is

$$
\begin{aligned}
&S_E(\psi_\xi(x,t);\lambda,\ldots)\\
&\quad= \int dt\, d^3x\ \ \psi^*(\xi x,\xi^2 t)\left(\partial_t - \frac{\nabla_x^2}{2m}\right)\psi(\xi x,\xi^2 t) + \frac{\lambda}{8m^2}(\psi^\dagger(\xi x,\xi^2 t)\psi(\xi x,\xi^2 t))^2 + \ldots\\
&\quad= \int dt'\, d^3x'\, \xi^{-3}\psi^*(x',t')\left(\partial_{t'} - \frac{\nabla_{x'}^2}{2m}\right)\psi(x',t') + \xi^{-5}\frac{\lambda}{8m^2}(\psi^\dagger(x',t')\psi(x',t'))^2 + \ldots\\
&\quad= S_E(\psi'(x,t);\xi\lambda,\ldots)\ .
\end{aligned}
\tag{15}
$$

where $\psi'(x) \equiv \xi^{-3/2}\psi(x)$. Thus in this case we find

$$
[x] = -1\ ,\quad [t] = -2\ ,\quad [\psi] = \frac{3}{2},\quad [\lambda] = -1\ . \tag{16}
$$

We see then that the ϕ^4 interaction, which was marginal for relativistic scalars, becomes the irrelevant $|\psi|^4$ interaction for nonrelativistic particles. In one of the problems I have provided, you are to show that that the $\lambda(\psi^\dagger\psi)^2$ interaction corresponds to a δ-function potential between two particles, and that the scaling properties we have derived make sense from the point of view of the Schrödinger equation.

Nonrelativistic EFT's are commonly used to describe nonrelativistic interactions between atoms or nuclei; for analyzing bound states such as positronium in QED (the EFT is referred to as "NRQED", for nonrelativistic QED); and for heavy quarkonia—bound states of a heavy quark-antiquark pair—in NRQCD. Such theories are more complicated that what I just described due to the interactions of light particles, such as the photon in NRQED and the gluon in NRQCD.

1.3 HQET

A somewhat different EFT is encountered in the analysis of hadrons containing a c or b quark whose mass is greater than the QCD scale Λ_{QCD}, which characterizes the binding energy of hadrons. While the heavy quark is nonrelativistic in the hadron rest frame, such hadrons can also include light quarks (u, d, s) which are not. Isgur and Wise noticed that there was a symmetry between mesons with a b quark, and mesons with a c quark in the limit that both were heavy, but with unequal masses. That there should be a symmetry is easy to see: as far as the light degrees of freedom are concerned, the heavy quark just looks like a static color source, pretty much independent of its (heavy) mass. Therefore the light quarks in a D meson and those in a B meson will have the same wavefunction, up to $1/M$ corrections. The system seems ideal for an EFT treatment, with Λ_{QCD}/M being the expansion parameter, where M is the heavy quark mass.

Since we are talking about boundstates of both heavy and light degrees of freedom, a nonrelativistic EFT is inadequate. A better EFT formalism was introduced by Georgi, who wrote the momentum of the heavy quark as

$$
p^\mu = mv^\mu + k^\mu\ , \tag{17}
$$

where m is the heavy quark mass, $v\mu$ is the 4-velocity of the heavy quark satisfying $v_\mu v^\mu = 1$, and $k^\mu = O(\Lambda_{QCD}$ is the "residual momentum" characterizing how off-shell the heavy quark is. He noted that in the large m limit, the velocity v^μ is unchanged in strong interaction processes, which involve momentum transfer of $O(\Lambda_{QCD})$. Therefore he introduced a different field for each velocity v:

$$h_v(x) = e^{im\not{v} v_\mu x^\mu} q(x) \ , \tag{18}$$

where $q(x)$ is the usual Dirac spinor for the heavy quark. Note that for $v = (1,0,0,0)$ the exponential prefactor is just $e^{imt\gamma_0}$, so the above transformation eliminates the fast time behavior for both particles and antiparticles at the same time. Then h_v can be decomposed into spinors h_v^+ which annihilates heavy quarks with velocity v, and h_v^- which creates heavy antiquarks with velocity v, via the projection operators

$$h_v^\pm = \left(\frac{1 \pm \not{v}}{2}\right) h_v \ . \tag{19}$$

One can then drop operators with interactions between h_v^+ and h_v^- as pair creation and annihilation is beyond the purview of the EFT. The free kinetic term then looks like

$$\mathcal{L}_0 = i\overline{h_v^+}\not{\partial} h_v^+ + i\overline{h_v^-}\not{\partial} h_v^- = i\overline{h_v}\not{v} v_\mu \partial^\mu h_v \ . \tag{20}$$

One great virtue of this EFT is that b and c quarks now look similar, despite the difference of the quark masses, and one can easily see the Isgur-Wise symmetry between them. HQET is used extensively in a Λ_{QCD}/m expansion to analyze heavy meson phenomenology which is important for determining the CKM angles of the standard model.

1.4 Some qualitative applications

This has all been rather formal. How are the scaling properties we have derived reflected in physical applications? I will consider here several qualitative examples.

1.4.1 Fermi's effective theory of the weak interactions

The term "weak interactions" refers in general to any interaction mediated by the $W^\pm$ or Z^0 bosons, whose masses are approximately 80 GeV and 91 GeV respectively. The currents they couple to are called "charged currents" and $neutral currents$ respectively. The charged currents are given by[3]

$$J_\pm^\mu = \frac{j_1^\mu \mp i\, j_2^\mu}{\sqrt{2}} \ , \tag{21}$$

where

$$j_a^\mu = \frac{e}{\sin\theta_w} \sum_\psi \overline{\psi}\gamma^\mu \left(\frac{1-\gamma_5}{2}\right)\frac{\tau_a}{2}\psi \ , a = 1,2,3 \ , \tag{22}$$

[3]Careful with factors of 2! I give here the currents that the $W^\pm$ and Z boson couple to; however, weak currents were historically defined to be twice these expressions, long before the standard model was written down.

with τ_a being the first two Pauli matrices and $g_2 \equiv e/\sin\theta_w$ being the $SU(2)$ gauge coupling (written in terms of the electromagnetic coupling e and the weak angle θ_w). The fermions ψ participating in the charged current interaction are the leptons

$$\psi = \begin{pmatrix} \nu_e \\ e \end{pmatrix} , \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} , \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} , \tag{23}$$

and the quarks

$$\psi = \begin{pmatrix} u \\ d' \end{pmatrix} , \quad \begin{pmatrix} c \\ s' \end{pmatrix} , \quad \begin{pmatrix} t \\ b' \end{pmatrix} , \tag{24}$$

with the "flavor eigenstates" d', s' and b' being related to the mass eigenstates d, s and b by the unitary Cabibbo-Kobayashi-Maskawa (CKM) matrix:

$$q'_i = V_{ij} q_j \ . \tag{25}$$

The elements of the CKM matrix are named after which quarks they couple through the charged current, namely $V_{11} \equiv V_{ud}$, $V_{12} \equiv V_{us}$, $V_{21} \equiv V_{cd}$, etc.

The Z^0 boson has a mass $M_Z = M_W \cos\theta_w$ and couples to the current

$$J^\mu_Z = \frac{e}{\sin\theta_w \, \cos\theta_w} \left(j_3 - \sin^2\theta_w j_{\rm em} \right) \tag{26}$$

where $j_{\rm em}$ is the electromagnetic current, where the neutrinos, charged leptons, up-type quarks and down-type quarks have $Q_{\rm em} = 0, -1, \frac{2}{3}$ and $-\frac{1}{3}$ respectively.

For many processes the dominant weak interaction is given by the tree level exchange of a W or Z boson. If the process is at low energy (where the momentum exchanged in any channel satisfies $p^2 \ll M_W^2$), then the gauge boson propagators may be approximated by a constant, by Taylor expanding in p^2/M^2

$$\frac{1}{p^2 - M^2} = -\frac{1}{M^2} + \frac{p^2}{M^4} + \ldots \tag{27}$$

and keeping only the leading term. Since the Fourier transform of a constant is a δ function, the weak boson exchange gives rise to a pointlike current-current interaction:

$$\mathcal{L}^{\rm weak}_{\rm eff} = 8\frac{G_F}{\sqrt{2}} \left(J^\mu_+ J_{-\mu} + \frac{1}{2} J^\mu_Z J_{Z\mu} \right) , \quad G_F = \frac{\sqrt{2}e^2}{8\sin^2\theta_w M_W^2} = 1.166 \times 10^{-5} \ \mathrm{GeV}^2 \tag{28}$$

The charged current part, written in terms of leptons and nucleons instead of leptons and quarks, was postulated by Fermi to explain neutron decay. Neutral currents were proposed in the 60's and discovered in the 70's. The relation between G_F and M_W is derived by "matching" — requiring that the two processes in Fig. 2 give the same S-matrix elements.

Since neutrinos carry no electric or color charge, in the standard model all of their low energy interactions are contained in $\mathcal{L}^{\rm weak}_{\rm eff}$ in eq. (28). Thus the neutrino cross-section σ must be proportional to G^2 which has dimension -4. But a cross section has dimensions of area, or mass dimension -2. If the scattering process of interest is relativistic, then the only

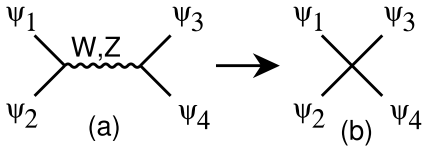

Figure 2: *(a) Tree level W and Z exchange between four fermions. (b) The effective vertex in the low energy effective theory (Fermi interaction).*

other scale around is the center of mass energy $\sqrt{s}$. Therefore on dimensional grounds, the cross-section must scale with energy as

$$\sigma_\nu \simeq G^2 s \ . \tag{29}$$

This explains why low energy neutrinos are so hard to detect, and the weak interactions are weak; they won't be at the LHC.

1.4.2 The blue sky

Consider the question of why the sky is blue. More precisely, consider the problem of low energy light scattering from neutral atoms in their ground state, where by "low energy" I mean that the photon energy E_γ is much smaller than the excitation energy ΔE of the atom, which is of course much smaller than its inverse size or mass:

$$E_\gamma \ll \Delta E \ll a_0^{-1} \ll M_{atom} \ .$$

Thus the process is necessarily elastic scattering, and to a good approximation we can ignore that the atom recoils, treating it as infinitely heavy. Let's construct an "effective Lagrangian" to describe this process. This means that we are going to write down a Lagrangian with all interactions describing elastic photon-atom scattering that are allowed by the symmetries of the world — namely Lorentz invariance and gauge invariance. Photons are described by a field A_μ which creates and destroys photons; a gauge invariant object constructed from A_μ is the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The atomic field is defined as ϕ_v, where ϕ_v destroys an atom with four-velocity v_μ (satisfying $v_\mu v^\mu = 1$, with $v_\mu = (1,0,0,0)$ in the rest-frame of the atom), while $\phi_v^\dagger$ creates an atom with four-velocity v_μ. So what is the most general form for $\mathcal{L}_{eff}$? Since the atom is electrically neutral, gauge invariance implies that ϕ can only be coupled to $F_{\mu\nu}$ and not directly to A_μ. So $\mathcal{L}_{eff}$ is comprised of all local, Hermitian monomials in $\phi_v^\dagger \phi_v$, $F_{\mu\nu}$, v_μ, and ∂_μ. Certain combinations we needn't consider for the problem at hand — for example $\partial_\mu F^{\mu\nu} = 0$ for radiation (by Maxwell's equations); also, if we define the energy of the atom at rest in it's ground state to be zero, then $v^\mu \partial_\mu \phi = 0$, since $v_\mu = (1,0,0,0)$ in the rest frame, where $\partial_t \phi = 0$. Similarly, $\partial_\mu \partial^\mu \phi = 0$. Thus we are led to consider the Lagrangian

$$\mathcal{L}_{eff} \quad = \quad c_1 \phi_v^\dagger \phi_v F_{\mu\nu} F^{\mu\nu} + c_2 \phi_v^\dagger \phi_v v^\alpha F_{\alpha\mu} v_\beta F^{\beta\mu}$$

$$+c_3\phi_v^\dagger\phi_v(v^\alpha\partial_\alpha)F_{\mu\nu}F^{\mu\nu}+\ldots \tag{30}$$

The above expression involves an infinite number of operators and an infinite number of unknown coefficients! Nevertheless, dimensional analysis allows us to identify the leading contribution to low energy scattering of light by neutral atoms. It is straightforward to figure out that

$$[\partial_\mu]=1\ ,\qquad [F_{\mu\nu}]=2\ ,\qquad [\phi]=\frac{3}{2}\ .$$

The first follows from the fact that ∂_μ has the dimension of 1/length. The second is easily determined by noting that the Maxwell Lagrangian is $\mathcal{L}_M=-\frac{1}{4}F^2$, and that $[\mathcal{L}]=4$. Finally ϕ is determined by writing a state with no atom as $|0\rangle$, and one atom as $|A\rangle$, where $\phi^\dagger(x)|0\rangle=\Psi_A(x)|A\rangle$, with $\Psi_A(x)$ being the normalized atomic wavefunction and $\langle 0|0\rangle=\langle A|A\rangle=1$. Since $\int d^3x\,|\Psi_A|^2=1$, it follows that $[\phi]=3/2$.

Since the effective Lagrangian has dimension 4, the coefficients c_1, c_2 etc. also have dimensions. It is easy to see that they all have negative mass dimensions:

$$[c_1]=[c_2]=-3\ ,\qquad [c_3]=-4$$

and that operators involving higher powers of $\partial\cdot v$ would have coefficients of even more negative dimension. It is crucial to note that these dimensions must be made from dimensionful parameters describing the atomic system — namely its size r_0 and the energy gap δE between the ground state and the excited states. The other dimensionful quantity, E_γ, is explicitly represented by the derivatives ∂_μ acting on the photon field. Thus for $E_\gamma\ll\Delta E, r_0^{-1}$ the dominant effect is going to be from the operator in $\mathcal{L}_{eff}$ which has the *lowest* dimension. There are in fact two leading operators, the first two in eq. eq. (30), both of dimension 7. Thus low energy scattering is dominated by these two operators, and we need only compute c_1 and c_2.

What are the sizes of the coefficients? To do a careful analysis one needs to go back to the full Hamiltonian for the atom in question interacting with light, and "match" the full theory to the effective theory. We will discuss this process of matching later, but for now we will just estimate the sizes of the c_i coefficients. We first note that extremely low energy photons cannot probe the internal structure of the atom, and so the cross-section ought to be classical, only depending on the size of the scatterer. Since such low energy scattering can be described entirely in terms of the coefficients c_1 and c_2, we conclude that

$$c_1\simeq c_2\simeq r_0^3\ .$$

The effective Lagrangian for low energy scattering of light is therefore

$$\mathcal{L}_{eff}=r_0^3\left(a_1\phi_v^\dagger\phi_v F_{\mu\nu}F^{\mu\nu}+a_2\phi_v^\dagger\phi_v v^\alpha F_{\alpha\mu}v_\beta F^{\beta\mu}\right) \tag{31}$$

where a_1 and a_2 are dimensionless, and expected to be $\mathcal{O}(1)$. The cross-section (which goes as the amplitude squared) must therefore be proportional to r_0^6. But a cross section σ has dimensions of area, or $[\sigma]=-2$, while $[r_0^6]=-6$. Therefore the cross section must be proportional to

$$\sigma\propto E_\gamma^4 r_0^6\ , \tag{32}$$

growing like the fourth power of the photon energy. Thus blue light is scattered more strongly than red, and the sky looks blue.

Is the expression eq. (32) valid for arbitrarily high energy? No, because we left out terms in the effective Lagrangian we used. To understand the size of corrections to eq. (32) we need to know the size of the c_3 operator (and the rest we ignored). Since $[c_3] = -4$, we expect the effect of the c_3 operator on the scattering amplitude to be smaller than the leading effects by a factor of E_γ/Λ, where Λ is some energy scale. But does Λ equal M_{atom}, $r_0^{-1} \sim \alpha m_e$ or $\Delta E \sim \alpha^2 m_e$? The latter is the smallest scale and hence the most important. We expect our approximations to break down as $E_\gamma \to \Delta E$ since for such energies the photon can excite the atom. Hence we predict

$$\sigma \propto E_\gamma^4 r_0^6 \left(1 + \mathcal{O}(E_\gamma/\Delta E)\right) . \tag{33}$$

The Rayleigh scattering formula ought to work pretty well for blue light, but not very far into the ultraviolet. Note that eq. eq. (33) contains a lot of physics even though we did very little work. More work is needed to compute the constant of proportionality.

1.4.3 The binding energy of charmonium in nuclei

Closely related to the above example is the calculation of the binding energy of charmonium (a $\bar{c}c$ bound state, where c is the charm quark) to nuclei. In the limit that the charm quark mass m_c is very heavy, the charmonium meson can be thought of as a Coulomb bound state, with size $\sim \alpha_s(m_c)m_c$, where $\alpha_s(m_c)$ is a small number (more on this later). When inserted in a nucleus, it will interact with the nucleons by exchanging gluons with nearby quarks. Typical momenta for gluons in a nucleus is set by the QCD scale $\Lambda_{QCD} \simeq 200$ MeV. For large m_c then, the wavelength of gluons will be much larger than the size of the charmonium meson, and so the relevant interaction is the gluon-charmonium analogue of photon-atom scattering considered above. The effective Lagrangian is just given by eq. (31), where ϕ now destroys charmonium mesons, and $F_{\mu\nu}$ is replaced by $G^a_{\mu\nu}$, the field strength for gluons of type $a = 1, \ldots, 8$. The coefficients $a_{1,2}$ may be computed from QCD. To compute the binding energy of charmonium we need to compute the matrix element

$$\langle N, \bar{c}c| \int d^3x\, \phi^\dagger \phi G^a_{\mu\nu} G^{a\,\mu\nu} |N, \bar{c}c\rangle$$

(as well as the matrix element of the other operator in eq. (31), which we do not know how to do precisely since the system is strongly interacting. We can estimate its size by dimensional analysis though, getting

$$E_B \sim r_0^3 \Lambda_{QCD}^4 \simeq \frac{\Lambda_{QCD}^4}{(\alpha_s m_c)^3} .$$

1.5 Problems

I.1) Redo the analysis of the relativistic scalar field theory for arbitrary dimension d. What is striking about the result for $d = 2$? In what dimension is ϕ^3 a marginal operator?

I.2) One defines the "critical dimension" d_c for an operator to be the spacetime dimension for which that operator is marginal. How will that operator behave in dimensions d when $d > d_c$ or $d < d_c$? In a theory of interacting relativistic scalars, Dirac fermions, and gauge bosons, determine the critical dimension for the following operators:

1. A gauge coupling to either a fermion or a boson through the covariant derivative in the kinetic term;
2. A Yukawa interaction, $\phi\overline{\psi}\psi$;
3. An anomalous magnetic moment coupling $\overline{\psi}\sigma_{\mu\nu}F^{\mu\nu}\psi$ for a fermion;
4. A four fermion interaction, $(\overline{\psi}\psi)^2$.

I.3) The Lagrangian eq. (12) is written as a field theory (second quantized form). Translating into first quantized form, show that the interaction is equivalent to a $\delta^d(r_1 - r_2)$ potential between two particles in d spatial dimensions, and relate the strength of the potential to λ. (A simple way to do this is to match the Born approximation (tree-level) amplitudes). What does the classification of the interaction as relevant, marginal or irrelevant correspond to when solving the Shrödinger equation in d dimensions with a δ-function potential? What is the critical dimension d for a $\delta^d(r_1 - r_2)$ potential? (Hint: you can think of the δ-function as being a limit of a sequence of square well potential that get deeper and narrower in such a way that the spatial integral of the potential is kept constant. How would the depth vs. width of a square well have to scale if you wanted to keep the physics constant, e.g. the energy of a boundstate, or the scattering length?)

I.4) Perform a scaling analysis for a relativistic Dirac fermion with mass m in d dimensions and a $(\overline{\psi}\psi)^2$ interaction. How does the scaling change in the nonrelativistic limit? Compare with ϕ^4 interaction in the bosonic case; is your answer consistent with the fact that the Schrödinger equation for nonrelativistic particles is the same whether they be fermions or bosons?

I.5) Derive eq. (28) and the relation between G_F and M_W by requiring that the scattering amplitudes calculated from the full and effective theories (Fig. 2) match.

2 Loops, symmetries, and matching

Up to now we have ignored quantum corrections in our effective theory. A Lagrangian such as eq. (1) is what used to be termed a "nonrenormalizable" theory, and to be shunned. The problem was that the theory needs an infinite number of counterterms to subtract all infinities, and was thought to be unpredictive. In contrast, a "renormalizable" theory contained only marginal and relevant operators, and needed only a finite number of counterterms, one per marginal or relevant operator allowed by the symmetries. A "superrenormalizable" theory contained only relevant operators, and was finite beyond a certain order in perturbation theory. However Wilson changed the view of renormalization. In a perturbative theory, irrelevant operators are renormalized, but stay irrelevant. On the other hand, the coefficients of relevant operators are renormalized to take on values proportional to powers of the cutoff, unless forbidden by symmetry. Thus in Wilson's view the relevant operators are the problem, since giving them small coefficients requires

2.1 Quantum corrections to scaling

We now turn to quantum corrections in an effective theory, such as eq. (1). It is evident that when inserted into loops, the different operators can renormalize each other. For example, the operator $c_1\phi^6/\Lambda^2$ can shift the $\lambda\phi^4$ interaction at one-loop by

$$\Delta\lambda \sim \frac{c_1}{(4\pi)^2} \ . \tag{34}$$

Here I have estimated a factor of $1/(4\pi)^2$ from the loop, and have noted that the quadratic divergence from the scalar loop is cut off at $p = \Lambda$, contributing a factor of Λ^2 which cancels the $1/\Lambda^2$ in front of the ϕ^6 operator. I began by assuming that all the dimensionless couplings in the theory were small, so the above correction is perturbative. That does not mean that $\Delta\lambda/\lambda \ll 1$, however, because I may have specified a very small λ. What we see here is that the "natural size" for λ is something at least as big as $\frac{c_1}{(4\pi)^2}$. I can of course choose the bare λ to nearly cancel against $\Delta\lambda$ in order to have a small physical ϕ^4 coupling, but that will involve fine tuning.

Similarly, at one loop the ϕ^4 interaction can multiplicatively renormalize the ϕ^6 interaction, yielding

$$\Delta c_1 \sim \frac{c_1\lambda}{(4\pi)^2}\ln\Lambda \ . \tag{35}$$

The form of the renormalizations follow simply from dimensional analysis. Evidently the radiative contributions to the dimensional couplings are all $\ll 1$, provided I start with small tree level couplings. However, there are a couple of exceptions.

2.1.1 Relevant operators and naturalness

The first has to do with the scalar mass term (or relevant operators in general). The mass term receives corrections of the form

$$\Delta m^2 \sim \left(\frac{\lambda}{(4\pi)^2} + \frac{c_1}{(4\pi)^4} + \dots \right) \Lambda^2 \ . \tag{36}$$

This correction is very big compared to m^2 even though the coupling constants are small, because of the factor of Λ^2. This is called an *additive* renormalization. You see, I have cheated in eq. (1): whereas all interactions aside from the mass term involved dimensionless couplings times the appropriate power of Λ, I wrote the mass term as $m^2\phi^2$. I should have written it as $c_{-2}\Lambda^2\phi^2$, where c_{-2} is a dimensionless coupling. Then I can rewrite the above equation as

$$\Delta c_{-2} \sim \left(\frac{\lambda}{(4\pi)^2} + \frac{c_1}{(4\pi)^4} + \dots \right) . \tag{37}$$

This tells us that we cannot have $m \ll \Lambda$ for this scalar without fine tuning, unless all of the particles interactions are extremely weak. Or: turn it around — the Higgs boson in the standard model, for example, has a ϕ^4 coupling of size $\lambda \simeq M_H^2/(M_W^2/g^2)$ where $g = e/\sin\theta_w \simeq 1$ is the $SU(2)$ gauge coupling. The theory will have to be fine-tuned if $\Delta M_H^2 \gg M_H^2$, or equivalently if $\Lambda \gg 4\pi M_W/g \simeq 1$ TeV. Of course, this doesn't mean that there can be no momenta above 1 TeV — rather, the cutoff in the effetive theory is simple the scale of short-range physics that has been omitted from the effective theory. So this suggests that the LHC should see new physics above a TeV...if you believe that nature doesn't like a finely tuned theory!

In contrast to scalar masses, fermion masses are not fine-tuned. At first sight, this is surprising, since a fermion mass term is $m\overline{\psi}\psi$, and $[\overline{\psi}\psi] = 3$. Thus it also looks like a relevant operator and one might expect $\Delta m \sim \frac{g^2}{(4\pi)^2}\Lambda$, where g is a gauge or Yukawa coupling. However, the kinetic term for the fermions obeys a chiral symmetry, under which $\psi \to e^{i\alpha\gamma_5}\psi$, which is broken by the fermion mass. This means that if $m = 0$, and there are no other interactions which violate chiral symmetry, then there can be no radiative corrections to the fermion mass at all. It follows then that if $m \neq 0$ is the only chiral symmetry violating operator in the theory, then radiative corrections to m must be proportional to m: $\Delta m \propto m$. On dimensional grounds then, the dependence of the radiative corrections to the fermion mass on the cutoff Λ can be at most logarithmic. Similar statements can be made about anomalous magnetic moment operators, since $\overline{\psi}\sigma^{\mu\nu}\psi$ also violates chiral symmetry.

This distinction between log and power law dependence of radiative corrections on Λ is not a small one. If we were to take $\Lambda = M_{pl} = 10^{19}$ GeV and $m = 100$ GeV, then power law divergences in the scalar mass imply a fine tuning of one part in $\frac{\lambda^2}{(4\pi)^2}m^2/\Lambda^2 = 10^{-34}\frac{\lambda^2}{(4\pi)^2}$ which is tiny unless the particle is essentially non-interacting ($\lambda = 0$); in contrast, the log corrections to a fermion mass are of size $\frac{\alpha}{4\pi}\log 10^{17} = 40\frac{\alpha}{4\pi}$, which is typically $O(1)$.

Similar arguments can be applied to the most relevant of all operators: the vacuum energy density, a constant term in the Lagrangian. The natural size of the vacuum energy is $\mathcal{E}_0 = \Lambda^4(\lambda/(4\pi)^4 + \dots$ The actual vacuum energy (cosmological constant), as measured in cosmology, is $\mathcal{E}_0 \simeq (10^{-3}\ \mathrm{eV})^4$, *much* lower than the fourth power of any sensible cutoff in the standard model. Apparently $\mathcal{E}_0$ is fine tuned by at least 60 orders of magnitude ($(1\ \mathrm{TeV}/10^{-3}\ \mathrm{eV})^4$)! Maybe Nature doesn't mind fine-tuning so much? (More on that later).

2.1.2 Logarithmic corrections

Further attention is warranted for the logarithmic radiative corrections, such as $\ln\Lambda/\mu$. They are especially interesting because they depend on the IR scale μ, which means that if one subtracts a radiative correction at scale μ from one at scale μ', there is a finite difference proportional to $\ln\mu'/\mu$. This is real physics, involving only the light degrees of freedom in the effective theory, and which cannot be absorbed into the redefinition of some local operator (in contrast, power law divergences can be, and so their values are scheme dependent...MS, for example, power law divergences all vanish!). In cases where $\alpha/4\pi\ln\mu'/\mu$ is large, the logs can be resummed using the renormalization group. Such corrections are ubiquitous, and occur for basically every operator in the theory. You can think of these logarithms as changing the scaling dimension of an operator. Consider a four fermion operator, such as a flavor changing interaction in the effective theory of the weak interactions. This is a dimension 6 operator, and so will appear suppressed by $1/\Lambda^2$,

$$\frac{c}{\Lambda^2}(\overline{\psi}\psi)^2\ . \tag{38}$$

Now suppose this operator receives a logarithmic renormalization of the form $\Delta c = -c\frac{\alpha}{4\pi}\ln\mu/\Lambda$, where α is some other coupling in the theory. For example, a correction like this could arise from a one-loop graph where the four fermion interaction is dressed with a gluon running between two fermion propagators. In the effective action then one would have the corrected coefficient

$$\frac{c(1-\frac{\alpha}{4\pi}\ln\mu/\Lambda)}{\Lambda^2}(\overline{\psi}\psi)^2\ . \tag{39}$$

Since I am assuming that perturbation theory in α is valid, to $O(\alpha)$ we can rewrite

$$\frac{c\,(1-\frac{\alpha}{4\pi}\ln\mu/\Lambda)}{\Lambda^2} = \frac{c\,e^{-\frac{\alpha}{4\pi}\ln\mu/\Lambda}}{\Lambda^2} = c\,\frac{1}{\Lambda^2}\left(\frac{\mu}{\Lambda}\right)^{-\alpha/4\pi}\ . \tag{40}$$

To one-loop order then, we see (by counting powers of Λ) that this four fermion operator scales not with scaling dimension 6, but rather $(6-\frac{\alpha}{4\pi})$, making it more important in the IR than naively expected. Note that this analysis assumes the validity of perturbation theory in α.

Quantum corrections to the scaling dimension of irrelevant operators, such as sketched above, can be important to include when predicting the rate of rare processes. An example would be in the study of flavor changing hadronic interactions such as $b\to s\gamma$, where one would like to uncover new weak-scale physics (such as supersymmetry) hiding behind the standard model.

It is when considering marginal operators that quantum corrections to the scaling dimension are particularly dramatic, as they make the operator either become relevant (strong in the IR, as in the case of the QCD coupling, or that of any nonabelian gauge theory with not too many matter fields) or irrelevant (as is the case with QED, ϕ^4 and Yukawa interactions). Hopefully you have encountered the renormalization group and β functions in a quantum field theory class; I will not review them here. However, I would like to demystify asymptotic freedom a bit by showing how it occurs in a mundane quantum mechanics problem, without the added complications inherent in a relativistic field theory.

Consider a particle in d spatial dimensions in a $\delta^d(r)$ potential with no angular momentum (for $d > 1$). In a second quantized language, a δ-function potential appears as a $(\lambda/8m^2)\left(\psi^\dagger\psi\right)^2$ interaction as in eq. (12). Generalizing yesterday's scaling arguments for arbitrary d one finds the scaling dimension of the interaction $[\left(\psi^\dagger\psi\right)^2] = (d-2)$. This means that the critical dimension is $d = 2$, and that the interaction is relevant in $d = 1$ and irrelevant in $d = 3$.

Let's see that directly from the Schrodinger equation:

$$-\nabla^2\Psi - g\delta^d(\mathbf{r})\Psi = 2ME\Psi \ . \tag{41}$$

On rescaling $\mathbf{r}$, we know that ∇^2 scales like $1/r^2$, while $\delta^d(\mathbf{r})$ scales like $1/r^d$, since $\int d^d\mathbf{r}\,\delta^d(\mathbf{r}) = 1$. To better understand the meaning of scaling a δ-function, replace the δ function with a square barrier of height V_0 and radius r_0, satisfying $V_0 S_d(r_0) = 1$, where $S_d(r_0)$ is the volume of a sphere of radius r_0 in d dimensions. For an attractive δ function potential, one has a square well, instead of a square barrier. The finite size of r_0 regulates the δ-function, and $1/r_0$ plays the role that our momentum cutoff Λ played in prior discussion. Clearly $S_d(r_0) \propto r_0^d$, and so $V_0 \propto 1/r_0^d$. Now take $r_0 \to 0$, and compare the relative importance of the kinetic versus the potential potential energy in the Schrödinger equation. Clearly $d = 2$ is special, for only in $d = 2$ do the kinetic and potential terms in the Schrödinger equation scale the same way.

In $d = 1$, the kinetic term dominates for short wavelength, and so high energy scattering will resemble a free theory (no potential); on the other hand, long distance physics is dominated by the potential term, and we find a bound state $\Psi = exp(-g|x|/2)$ with binding energy $g^2/8M$ in the limit $r_0 \to 0$.

In $d = 3$, with the scaling law we chose for V_0 we get nonsense if the interaction is attractive! The potential well has depth $V_0 \propto 1/r_0^3$ and so potential energy dominates at short distance, and we get an infinite number of bound states. Quite appropriate for an irrelevant operator: it is sensitive to short distance physics. In the second quantized form, we can see that the energy is not bounded from below. This problem is not encountered with a repulsive $\delta^3(\mathbf{r})$ potential.

Since we will later be talking about an effective theory for nucleon-nucleon scattering in $d = 3$, it is useful to point out that the $d = 3$ theory with an attractive interaction can be forced to make sense by taking liberty with what we mean by a δ function interaction, and by changing our scaling law for V_0. In particular, if we require that be a bound state at some fixed energy as we take $r_0 \to 0$, we find that we must take V_0 to scale as $1/r_0^2$, and not $1/r_0^3$. This means we are replacing the $g\delta^3(\mathbf{r})$ potential by a regulated potential $g' r_0 \delta^3_{r_0}(\mathbf{r})$. This is equivalent to saying that our "bare" coupling $g' r_0$ goes to zero as we remove the regulator (that is, take $r_0 \to 0$) for fixed renormalized coupling g', whose value is determined by the renormalization condition that we find a bound state with a specified energy.

The $d = 2$ case is particularly interesting we have seen that $d = 2$ is the critical dimension for a δ-function potential. If we replace the $g\delta^{(2)}(\mathbf{r})$ in eq. (41) by a 2d square well of depth $V_0 = g/(\pi r_0^2)$ and solve the Schrödinger equation for a bound state with fixed binding energy (our renormalization condition) we find a solution in terms of Bessel functions which is nonsingular at the origin, bounded at $r = \infty$ and continuous at $r = r_0$ of the form

$$\Psi_< = J_0(pr) \ , \quad \Psi_> = \frac{J_0(pr_0)K_0(qr)}{K_0(qr_0)} \ , \qquad p \equiv \sqrt{-2ME} \ , \qquad q \equiv \sqrt{g/(\pi r_0^2) - p^2} \ , \tag{42}$$

subject to the condition that the first derivative is continuous at $r = r_0$:

$$\Psi'_<(r_0) = \Psi'_>(r_0) \ . \tag{43}$$

Assuming that the coupling g is small, we can expand and solve the above equation to linear order in g, and then expand that solution for small r_0. Doing this, I find

$$g = \frac{2\pi}{\frac{1}{2} - \gamma - \ln(pr_0/2)} + O(r_0) = \frac{2\pi}{\ln(1/r_0\Lambda)} + O(r_0) \ , \qquad \Lambda \equiv \frac{p}{2e^{1/2-\gamma}} \simeq 0.54\,p \tag{44}$$

where $\gamma = 0.5772\ldots$ is the Euler γ-function. Evidently, our bare g vanishes like an inverse logarithm of r_0 as $r_0 \to 0$. It also appears to blow up in the IR at $r_0 = 1/\Lambda$, but this behavior is not to be trusted since I employed perturbation theory to derive the result. This behavior closely resembles that of the QCD coupling constant, when one fixes the mass of some hadron, such as a glueball. The running QCD coupling vanishes like an inverse logarithm at short distance (asymptotic freedom), with Λ_{QCD} playing the role of Λ in the above equation. Note that if I replace r_0 by $1/\mu$ in the above expression, then $g(\mu)$ obeys the equation

$$\mu\frac{dg}{d\mu} = -\frac{g^2}{2\pi} \tag{45}$$

which is the renormalization group equation for the running coupling constant in this theory. The fact that the right hand side of the above equation is negative indicates asymptotic freedom. Apparently if we had analyzed a repulsive interaction ($g \to -g$), fixing something physical, such as the scattering length, then the RG equation would have given an asymptotically *unfree* theory...the coupling would get stronger in the UV, and one would not be able to take the $r_0 \to 0$ limit, as one would encounter a "Landau pole" as in QED — the coupling g would become infinite at finite r_0.

Hopefully these examples will convince you that the regularization, renormalization and running couplings encountered in quantum field theory have to do with the singular nature of local interactions, and have nothing to do with relativity, or the fact that relativistic quantum theories are many-body theories.

2.2 Integrating out massive fields and matching

One reason for using an effective field theory description is to sum up the logarithmic enhancements to irrelevant operators. As an example consider $\Delta B = 2$ processes in the

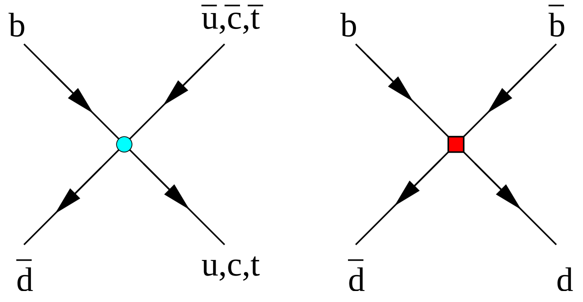

Figure 3: $\Delta B = 1$ *and* $\Delta B = 2$ *vertices in the effective theory.*

standard model, which occur at one loop and which contribute to $B - \overline{B}$ meson mixing. The effective theory contains contact interactions for both $\Delta B = 1$ and $\Delta B = 2$ processes (Fig. 3); both are dimension 6 operators, and will be multiplied by a dimensionless number to be determined, divided by the cutoff of the effective theory, which is M_W. To determine the coefficients is called "matching", and one does it in perturbation theory (a loop expansion).

First one calculates the tree diagram for $\Delta B = 1$ processes in the standard model, and matches onto the $\Delta B = 1$ contact interaction in the effective theory (Fig. 4). In doing so one only keeps the leading part of the tree diagram in a p^2/M_W^2 expansion. This is identical to how one would determine the relation between M_W and G_F in the Fermi theory.

Next one does 1-loop matching. It turns out we do not have to consider 1-loop contributions to the $\Delta B = 1$ operator for a leading order calculation of $\Delta B = 2$ processes. However, we do need to compute the coefficient of the $\Delta B = 2$ contact interaction at this order. In the standard model, we have the box diagram in Fig. 4, which must be equated to the sum of the loop diagram in the effective theory with two tree-level matched $\Delta B = 1$ vertices, plus the $\Delta B = 2$ contact interaction, thereby fixing the coefficient of the latter. One can do the matching with zero momentum flowing through the external propagators to leading order in the p^2/M_W^2 expansion. The diagram in the effective theory is divergent, and a renormalization scale μ must be chosen; it is convenient to choose $\mu = M_W$, the scale at which one is performing the matching so that $\ln(\mu/M_W)$ factors vanish..

Note that in the 1-loop matching, the loop diagram in the effective theory has the same light internal degrees of freedom as the full box diagram, and so will exactly reproduce the infrared physics. on the other hand, we have butchered the UV physics by replacing the W propagator with a pointlike vertex, and the $\Delta B = 2$ contact interaction is adjusted to fix that up.

Now that the $\Delta B = 2$ vertex is determined to one-loop accuracy, one can consider a gluon line dressing it. This is formally at the 2-loop level. However, if one wishes to take a matrix element of the $\Delta B = 2$ operator in the B meson state, gluon loops renormalized at $\mu = M_W$ will contribute factors of $\alpha_s(M_W) \ln M_W/M_B$, where the log is pretty big. Therefore the appropriate thing to do is to compute the anomalous dimension of the $\Delta B = 2$ operator due to one gluon exchange between quark legs, and run the coupling down to $\mu = M_B$. This procedure sums up powers of $\alpha_s(M_W) \ln M_W/M_B$ and leaves one

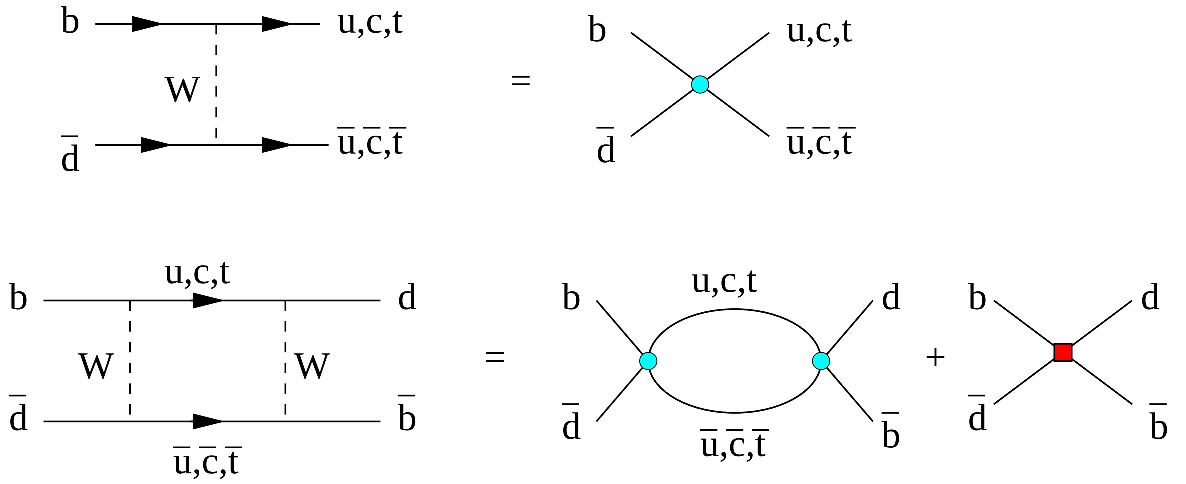

Figure 4: *At tree level one matches the $\Delta B = 1$ diagram from the standard model onto the $\Delta B = 1$ contact interaction of the effective theory. At one loop level one determines the $\Delta B = 2$ contact term of the effective theory by matching the $\Delta B = 2$ box diagram from the standard model onto the one-loop graph involving the two $\Delta B = 1$ vertices in the effective theory, plus the $\Delta B = 2$ contact term. The calculation requires specification of renormalization scale, conveniently taken to be $\mu = M_W$.*

with an expansion in $\alpha_s(M_B)$.

This whole procedure is possible because one has separated cleanly the short distance physics (W exchange) from the long distance physics (gluon exchange) by means of the effective theory. This procedure was first worked out by Gilman and Wise for the $\Delta S = 1$ Hamiltonian [1].

2.3 Power counting in a nonperturbative theory

We have seen that the logarithmic renormalizations are interesting in a perturbative theory, as they determine whether naively marginal operators become relevant or irrelevant when quantum corrections are included. However, in a strongly coupled, nonperturbative theory, there may in fact be no relation between the scaling dimensions of operators which were derived in the classical action, and the true scaling in the quantum theory. In general, when the couplings of the effective theory are $O(1)$ times the appropriate power of Λ, there will be a first order phase transition in the theory, and modes will will become heavy with masses $O(\Lambda)$, and the theory is of no use for describing physics below the cutoff. However, in some cases the theory will exhibit a second order phase transition, with the concomitant diverging correlation lengths, which can be interpreted in terms of particles whose masses become vanishingly small compared to the cutoff Λ as the couplings of the theory are tuned to the critical values. In this case the physical light degrees of freedom might be a composite of the original degrees of freedom, and the scaling dimension will be radically different than the naive prediction. For example, if a weakly coupled scalar emerges which is a bound state of fermion ψ and antifermion $\overline{\psi}$, then $[\overline{\psi}\psi] \simeq 1$, even the the perturbative scaling dimension is $[\overline{\psi}\psi] = 3$ (in $d = 4$ spacetime dimensions). In general, an EFT in terms of the constituent variables in a strongly coupled theory is not useful. Usually the best one can do is express the EFT in terms of the composite degrees of freedom, and write down the most general set of operators consistent with the symmetries of the underlying theory.

2.4 Symmetry

Symmetry is an invaluable tool in effective field theory. We saw, for example, that chiral symmetry makes a fermion mass term in four dimensions behave like a marginal operator, rather than a relevant one. Symmetries also have implications for what operators are allowed in the effective theory and what the light degrees of freedom are. There are several basic ways symmetries play a role, and so I will offer here a brief survey.

2.4.1 Gauge symmetries

Gauge symmetries are not really symmetries, they are constraints that allow one to describe forces with a redundancy of variables, allowing one to maintain manifest relativistic covariance. In general, gauge symmetries cannot be broken, and when constructing an effective theory to provide a low energy description of a more general theory, any gauge symmetry of the more complete theory will be inherited by the low energy theory. However, this can be done deviously. For example, compare QCD and the effective theory for pion physics (the chiral Lagrangian). Both manifestly respect electromagnetic gauge invariance. Both also respect $SU(3)_c$ gauge invariance, but in a more trivial way — the theory of pions only consists of color neutral objects, so saying that the theory is gauge invariant is rather trivial.

Less obvious might be what happens to gauge invariance when passing from the $SU(2) \times U(1)$ gauge theory of electroweak interactions, to Fermi's effective theory. The kinetic terms for quarks in the former involve gauge covariant derivatives, but not in Fermi's theory. However, to derive Fermi's theory from $SU(2) \times U(1)$ one actually has to "gauge fix" first...this doesn't change the physics, but makes you loose manifest gauge invariance. If you didn't gauge fix, there would be all sorts of spurious degrees of freedom in the effective theory. We put up with them in the $SU(2) \times U(1)$ theory in order to maintain relativistic covariance, but we do not need them in the low energy theory since there are no long range weak forces, and so we do without.

An even more exotic example was provided by Seiberg who showed that certain strongly coupled supersymmetric theories, where two theories with different gauge symmetries are shown to be identical to each other ("duals").

2.4.2 Exact and approximate global symmetries

The existence of a global symmetries (as opposed to gauged) means that there are charges which commute with the Hamiltonian. If the short distance theory possesses an exact symmetry, then the low energy EFT must as well, although it might be trivially realized in that all the light degrees of freedom are neutral under that symmetry (for example, at energy scales below the pion mass, an EFT for the standard model would consist of the electron, muon three flavors of neutrino, and the photon; baryon number is an exact symmetry in this theory, but none of the degrees of freedom carry baryon number.)

It often happens that a symmetry is not exact at short distance. If the breaking is in some sense small, then one can perturb in the symmetry breaking parameter in the low energy EFT, and still exploit the symmetry to constrain the allowed operators at any order in the expansion. There are several general cases to consider:

i. The symmetry could be softly broken: that is to say, it is broken by a relevant operator. Chiral symmetry in QCD is such an example, as it is a symmetry of the fermion kinetic terms but is broken by the fermion masses. Such breaking becomes more important as one scales to low energies. In chiral perturbation theory, for example, which is formulated at a scale $\Lambda \sim 1$ GeV, there is no sense in which chiral symmetry is approximate for the t b or c quarks, as their masses are larger than Λ; on the other hand, chiral symmetry is still quite good for the u and d quarks, whose masses are several MeV. For the strange quark with mass ~ 100 MeV, chiral symmetry is reasonably good. We will see in tomorrow's lecture how these masses are incorporated into the chiral Lagrangian.

ii. The symmetry could suffer small "hard" symmetry breaking due to a marginal operator. An example is the breaking of isospin in QCD by electromagnetism. The gauge coupling to quarks is dimension 4, but because $\alpha/4\pi$ is small, electromagnetism does not break isospin symmetry by much. For example,

$$\frac{m_{\pi^+}^2 - m_{\pi^0}^2}{m_\rho^2} \simeq 0.002 \sim \alpha/3\ . \tag{46}$$

(You will see later why the above ratio is a sensible one to take, rather than dividing by a pion mass, for example). If one is considering a very large span of length scales (for example, from the GUT scale at 10^{16} GeV down to Λ_{QCD}, one might need to run these hard symmetry breaking terms by means of the renormalization group, to resum the logarithmic we discussed above.

Note that symmetry breaking like this is only possible if there is no way for the marginal symmetry breaking operators to generate relevant symmetry breaking terms. In the above case of case of isospin breaking, QED radiative corrections *can* renormalize quark masses, but as argued above, chiral symmetry ensures that quark masses behave like marginal operators, and so electromagnetism and the u-d quark mass difference will play similar roles in isospin breaking in the chiral Lagrangian. However, consider a scalar field with a shift symmetry $\phi \to \phi + f$, which would forbid a scalar mass term. No suppose the symmetry is broken by a small $\lambda\phi^4$ interaction. This interaction will generate a mass term for the scalar at one loop, of size $\lambda/(4\pi)^2\Lambda^2\phi^2$. The moral is that if you break a symmetry with some operator, all possible symmetry breaking operators will be generically be radiatively generated, multiplied by coupling constants and the appropriate power of the cutoff Λ. Exceptions to this rule occur in supersymmetric theories.

iii. Explicit symmetry breaking could occur in irrelevant operators. This is peculiar: it implies that the symmetry is better at low energies than at high energies. In fact, one can discover symmetries in the low energy EFT that do not exist at all at high energy! These are called accidental symmetries, and I will say more about them below.

2.4.3 Spontaneously broken symmetries

Global symmetries can be spontaneously broken, which means that the ground state $|\Omega\rangle$ of the theory is not invariant under the symmetry transformation. If Q is the symmetry charge for a continuous symmetry (such as $U(1)$, as opposed to a discrete symmetry such as CP) then spontaneous symmetry breaking implies $Q|\Lambda\rangle \neq 0$, so that a symmetry rotation of the ground state produces a new state:

$$e^{i\alpha Q}|\Lambda\rangle = |\Lambda'\rangle \ . \tag{47}$$

The charge Q still commutes with the Hamiltonian, $[Q, H] = 0$, assuming it was an exact symmetry to begin with. Therefore the state $|\Lambda'\rangle$ is degenerate with the old one $|\Lambda\rangle$, and is also a perfectly fine ground state of H. Thus spontaneous symmetry breaking implies an infinitely degenerate manifold of ground states.

The universe as a whole will choose one of those ground states at random. However, there will be low energy excitations where the groundstate smoothly interpolates between $|\Lambda\rangle$ in one region of space and $|\Lambda'\rangle$ in another; these excitations must have vanishingly small energy at long wavelength as the groundstates $|\Lambda\rangle$ and $|\Lambda'\rangle$ are degenerate. Such excitations are called Goldstone bosons. It is important to realize that since $[Q, H] = 0$ even if there is spontaneous symmetry breaking, that means that the symmetry is still exact. It is just realized in a funny way, under which the Goldstone bosons shift, e.g, $\phi \to \phi + \alpha$. Symmetry multiplets are filled out with many-particle states now, involving massless Goldstone bosons.

If the symmetry is both explicitly and spontaneously broken, then $[Q, H] \neq 0$ and there is typically a unique ground state $|\Lambda\rangle$, but a manifold of states $|\Lambda'\rangle$ which are nearly degenerate with it. This means that there are massive excitations connecting the states; if the explicit symmetry breaking is small, then the excitations can be light and are called pseudo Goldstone bosons (or PGBs). Pions are thought to be PGBs; more on this tomorrow.

Goldstone bosons and PGBs often play a role in effective field theory for the obvious reason that they are naturally light degrees of freedom.

2.4.4 The standard model as an EFT: love those accidental symmetries

It is profitable to consider the standard model to be an EFT, and ask what operators can be added to it, consistent with the $SU(3) \times SU(2) \times U(1)$ gauge symmetry. Interestingly enough, there is only one relevant or marginal operators that could be added, and that is a $\theta \widetilde{G}_{\mu\nu} G^{\mu\nu}$ term for the gluons. This operator violates T and CP, and we know from lack of a fundamental electric dipole moments for the neutron that $\theta \lesssim 10^{-9}$ (where θ is defined in a basis where quark masses are real and positive). The absence of this operator in the standard model is called the "strong CP problem".

At dimension 5, a new contribution exists, of the form

$$\frac{\lambda_{ij}}{\Lambda}(\ell_i H^\dagger)(\ell_j H^\dagger) + \text{h.c.} \ , \tag{48}$$

where ℓ is the left-handed lepton doublet from family i, and H is the Higgs doublet. When the Higgs field is given a vev, this term gives a Majorana (lepton number violating) mass of size $m_\nu \sim (250\ \mathrm{GeV})^2/\Lambda$. Light neutrino masses are observed, in nature, and the existence of this dimension 5 operator can explain why they are so small: Λ is big. For example, with $\Lambda = 10^{14}$ GeV, $m_\nu \sim 0.6$ eV.

At dimension 6, one encounters a host of four fermion operators. The most interesting classes are those which lead to flavor changing neutral currents (FCNC), such as those contributing to $b \to s\gamma$, $K^0 \to \mu^+\mu^-$, and $\Delta S = 2$ operators; those which give rise to CP-violating electric dipole moments for quarks and leptons (to date unobserved); and those violating baryon number. The lack of evidence for any of these processes can be nicely explained in the standard model: the scale Λ where new physics kicks in must be very high. This is a hugely appealing feature of the standard model: even if baryon number is completely violated at the scale Λ, it becomes an accidental symmetry in the standard model simply because gauge invariance and the particle content restricts the theory so much that there are no relevant or marginal gauge invariant baryon number violating operators. Similarly for lepton number.

However, the standard model has a problem in the form of a relevant operator, the Higgs mass, which must be in the 100 GeV range for the model to work. As discussed earlier, the natural size for m_H^2 will be $\lambda/(4\pi)^2\Lambda^2$, and so raising Λ to explain the absence of rare processes makes the fine tuning of the Higgs mass worse. Introducing new physics at the scale $\Lambda = 1$ TeV (such as supersymmetry or technicolor), makes it possible to introduce symmetries which forbid the introduction of a Higgs mass term, solving the fine-tuning problem. However, now that Λ is low, it is no longer automatic that baryon number and lepton number should be well conserved, nor that FCNC should be absent.

The sociological pendulum has swung: since 't Hooft's emphasis on naturalness in the late 1970's physicists became obsessed with solving the naturalness problem, and then by destroying accidental symmetries, had to face difficult model building issues to solve problems which aren't problems in the standard model. Now people are questioning whether fine-tuning is bad at all, or whether only in a finely tuned universe would galaxies and a world like ours be possible. For example, if the cosmological constant were much bigger than it is, the universe would fly apart and galaxies would never form. So instead of asking: "is it unlikely that cosmological constant should be small?" (whose answer is presumably "Yes") one asks: " is it unlikely that the cosmological should be small in our universe, given that it has galaxies?", to which the answer is "No"). This goes under the rubric "anthropic principle" although the proponents are trying making it sound more respectable with the title "galactic principle". It is not as stupid as I am making it sound, but it would take more time than I have to convince you of that. I encourage you to read some of the literature on the subject if you are interested.

2.4.5 Quantum field theory on the lattice: more accidental symmetries

Accidental symmetries also play an important role in lattice field theory, where one formulates a field theory on a lattice spacetime, and computes properties numerically, approaching the continuum limit by sending the lattice spacing to zero. But why is it obvious that a hypercubic lattice will yield a continuum Lorentz invariant theory? The reason lattice field theory works is because of accidental symmetry: Operators on the lattice are constrained by gauge invariance and the hypercubic symmetry of the lattice. While it is possible to write down operators which are invariant under these symmetries, and which also violates Lorentz symmetry, such operators have high dimension and are not relevant. For example, if A_μ is a vector field, the operator $A_1A_2A_3A_4$ is hypercubic invariant, relevant and violates Lorentz symmetry; however, the only vector field in lattice QCD is the gauge potential, and such an operator is forbidden because it is not gauge invariant. Since irrelevant operators vanish in the continuum limit, Lorentz symmetry is achieved, even though it did not exist at short distance on the lattice.

The same principle has been recently exploited in constructing supersymmetric gauge theories on the lattice [2–7]. Supersymmetry is related to Lorentz symmetry, and one cannot have the full supersymmetry be exact on the lattice. The idea is to maintain enough exact supersymmetry,however, to ensure that operators that do not respect the full continuum supersymmetry are irrelevant.

Another example of accidental symmetry on the lattice is afforded by domain wall fermions. It was long known that there was a problem formulating lattice fermions invariant under a chiral symmetry, which makes it very difficult to simulate real QCD. However, I was able to show that on a five dimensional lattice one could formulate a theory of massive fermions without any chiral symmetry, which led to some massless fermions living on the four dimensional surface of the box [8]. It was therefore evident that the low energy effective theory for this system corresponded to a four dimensional world of massless fermions and an accidental chiral symmetry.

2.5 Problems

II.1) Consider a repulsive δ function interaction in $d = 2$ and $d = 3$. Regulate the δ function as a spherical barrier of height V_0 and radius r_0 inside a box of fixed radius R_0, and look at the lowest energy eigenvalue as $r_0/R_0 \to 0$, scaling V_0 appropriately. Compare your result with that of the free theory, $V_0 = 0$. Make sense of your answer given our analysis that $d = 2$ is the critical dimension for the δ-function potential for nonrelativistic particles.

II.2) Derive the RG equation eq. (45) by more conventional techniques by considering the one loop diagram for two particle scattering using dimensional regularization and an MSsubtraction scheme. You may find some of the formalism presented in the fourth lecture to be useful here, as well as the appendix B.

II.3) Write down a dimension 6 operator in the standard model which violates baryon number, and estimate the life of a proton in terms of Λ. If the life of a proton is $> 10^{34}$ years, how big does Λ have to be?

II.4) Show that any dimension 6 operator which violates baryon number preserves $(B - L)$, where B is baryon number and L is lepton number. A process that violates B but not L could contribute to oscillations between neutrons and anti-neutrons. What is the lowest dimension operator in the standard model which violates $B - L$ symmetry and can contribute to $n - \bar{n}$ oscillations? Roughly what is the oscillation rate as a function of Λ?

3 Chiral perturbation theory

3.1 Chiral symmetry in QCD

QCD is the accepted theory of the strong interactions. At large momentum transfer, as in deep inelastic scattering processes and the decays of heavy particles such as the Z, the theory is perturbative due to asymptotic freedom. The flip side is that in the infrared, the theory becomes nonperturbative. This good in the sense that we know that the light hadrons don't look at all like a collection of quarks weakly interacting via gluon exchange. But it does mean that QCD is not of much help in quantitatively understanding this phenomenology without resorting to lattice QCD and a computer. However, there does exist an effective field theory which is very powerful for treating analytically the interactions of the lightest hadrons, the pseudoscalar octet, consisting of the π, K, $\overline{K}$ and η.

The reason that the pseudo-scalar octet is lighter is because they are pseudo-Goldstone bosons (PGBs) that arise from the spontaneous breaking of an approximate symmetry in QCD.

Consider the QCD Lagrangian, keeping only the three lightest quarks, u, d and s:

$$\mathcal{L} = \sum_{i=1}^{3} \left(\bar{q}_i i\not{D} q_i - m_i \bar{q}_i q_i\right) - \frac{1}{2}\mathrm{Tr}G_{\mu\nu}G^{\mu\nu} \ , \tag{49}$$

where $D_\mu = \partial_\mu + igA_\mu$ is the covariant derivative, $A_\mu = A_\mu^a T_a$ are the eight gluon fields with T_a being $SU(3)$ generators in the 3 representation, and $G_{\mu\nu}$ being the gluon field strength. Note that if I write the kinetic term in terms of right-handed and left-handed quarks, projected out by $(1 \pm \gamma_5)/2$ respectively, then the kinetic term may be written as

$$\sum_i \bar{q}_i i\not{D} q_i = \sum_i \left(\bar{q}_{Li} i\not{D} q_{Li} + \bar{q}_{Ri} i\not{D} q_{Ri}\right) \ . \tag{50}$$

This term by itself evidently respects a $U(3)_L \times U(3)_R$ symmetry, where I rotate the three flavors of left-handed and right-handed quarks by independent unitary matrices. One combination of these transformations, the $U(1)_A$ transformation where $q_i \to e^{i\alpha\gamma_5} q_i$ is in fact not a symmetry of the quantum theory, due to anomalies; it is a symmetry of the action but not of the measure of the path integral. This leaves us with a $U(1)_V \times SU(3)_L \times SU(3)_R$ symmetry. The $U(1)_V$ is just baryon number, under which both left- and right-handed quarks of all flavors pick up a common phase. The remaining $SU(3)_L \times SU(3)_R$ symmetry, under which $q_{Li} \to L_{ij} q_{Lj}$ and $q_{Rj} \to R_{ij} q_{Rj}$, where R and L are independent $SU(3)$ matrices, is called "chiral symmetry".

$SU(3)_L \times SU(3)_R$ is not an exact symmetry of QCD, however. The quark mass terms may be written as

$$\sum_i m_i \bar{q}_i q_i = \sum_{i,j} \bar{q}_{Ri} M_{ij} q_{Lj} + h.c. \ , \qquad M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix} , \tag{51}$$

where the quark masses m_i are called "current masses", not to be confused with the much bigger constituent quark masses in the quark model. Since the mass term couples left- and right-handed quarks, it is not invariant under the full chiral symmetry. Several observations:

- Note that *if* the mass matrix M were a dynamical field, transforming under $SU(3)_L \times SU(3)_R$ as

$$M \to RML^\dagger \ , \tag{52}$$

then the Lagrangian *would* be chirally invariant. Thinking of the explicit breaking of chiral symmetry as being due to spontaneous breaking due to a field M which transforms as above makes it simple to understand how M must appear in the effective theory, which will have to be chirally invariant given the above transformation. This is called treating M as a "spurion".

- The symmetry is broken to the extent that $M \neq RML^\dagger$. Since m_u and m_d are much smaller than m_s, $SU(2)_L \times SU(2)_R$ is not broken as badly as $SU(3)_L \times SU(3)_R$;
- If all three quark masses were equal but nonzero, then there QCD would respect an exact $SU(3)_V \subset SU(3)_L \times SU(3)_R$ symmetry, where one sets $L = R$. This is the $SU(3)$ symmetry of Gell-Mann.
- Since $m_d - m_u$ is small, $SU(2)_V \subset SU(3)_V$, where $L = R$ and they act nontrivially only on the u and d quarks, is quite a good symmetry...also known as isospin symmetry.
- Independent vectorlike phase rotations of the three flavors of quarks are exact symmetries...these three $U(1)$ symmetries are linear combinations of baryon number, I_3 isospin symmetry, and Y (hypercharge). The latter two are violated by the weak interactions, but not by the strong or electromagnetic forces.

We know that this still is not the whole story though. An added complication is that the QCD vacuum spontaneously breaks the chiral $SU(3)_L \times SU(3)_R$ symmetry down to Gell-Mann's $SU(3)_V$ via the quark condensate:

$$\langle 0|\bar{q}_{Rj} q_{Li}|0\rangle = \Lambda^3 \delta_{ij} \ , \tag{53}$$

which transforms as a $(3, \bar{3})$ under $SU(3)_L \times SU(3)_R$. Here Λ has dimensions of mass. If one redefines the quark fields by a chiral transformation, the Kronecker δ-function above gets replaced by a general $SU(3)$ matrix,

$$\delta_{ij} \to (LR^\dagger)_{ij} \equiv \Sigma_{ij} \ . \tag{54}$$

If $L = R$ (an $SU(3)_V$ transformation), $\Sigma_{ij} = \delta_{ij}$ which shows that the condensate leaves unbroken the $SU(3)_V$ symmetry. For $L \neq R$, Σ_{ij} represents a different vacuum from eq. (53), and if it wasn't for the explicit breaking of $SU(3)_L \times SU(3)_R$ by quark masses in the QCD Lagrangian, these two different vacua would be degenerate. By Goldstone's theorem therefore, there would have to be eight exact Goldstone bosons — one for each of the eight broken generators — corresponding to long wavelength, spacetime dependent rotations of the condensate. We will parametrize these excitations by replacing

$$\Sigma \to \Sigma(x) \equiv e^{2i\boldsymbol{\pi}(x)/f} \ , \qquad \boldsymbol{\pi}(x) = \pi_a(x) T_a \tag{55}$$

where the T_a are the $SU(3)$ generators ($a = 1, \ldots, 8$) in the defining representation normalized to

$$\mathrm{Tr} T_a T_b = \frac{1}{2}\delta_{ab} \ , \tag{56}$$

f is a parameter with dimension of mass which we will relate to the pion decay constant f_π, and the π_a are eight mesons transforming as an octet under $SU(3)_V$. These bosons correspond to long wavelength excitations of the vacuum.

If you are somewhat overwhelmed by this amazing mix of symmetries that are gauged, global, exact, approximate, spontaneously broken and anomalous (and usually more than one of these attributes at the same time), rest assured that it took a decade and many physicists to sort it all out (the 1960's).

3.2 Quantum numbers of the meson octet

It is useful to use the basis for $SU(3)$ generators $T_a = \frac{1}{2}\lambda_a$, where λ_a are Gell Mann's eight matrices. The meson matrix $\boldsymbol{\pi} \equiv \pi_a T_a$ appearing in the exponent of Σ is a traceless 3×3 matrix. We know that under and $SU(3)_V$ transformation $L = R = V$,

$$\Sigma \to V\Sigma V^\dagger = e^{2iV\boldsymbol{\pi}V^\dagger/f} \ , \tag{57}$$

implying that under $SU(3)_V$ the mesons transform as an octet should, namely

$$\boldsymbol{\pi} \to V\boldsymbol{\pi}V^\dagger \ . \tag{58}$$

Then by restricting V to be an I_3 (T_3) or a Y (T_8) rotation we can read off the quantum numbers of each element of the $\boldsymbol{\pi}$ matrix and identify them with real particles:

$$\boldsymbol{\pi} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \tag{59}$$

An easy way to understand the normalization is to check that

$$\mathrm{Tr}(\boldsymbol{\pi}\boldsymbol{\pi}) = \frac{1}{2}\sum_a (\pi_a)^2 = \frac{1}{2}(\pi^0)^2 + \frac{1}{2}\eta^2 + \pi^+\pi^- + K^+K^- + K^0\overline{K}^0 \ . \tag{60}$$

3.3 The chiral Lagrangian

3.3.1 The leading term and the meson decay constant

We are now ready to write down the effective theory of excitations of the chiral condensate (the chiral Lagrangian), ignoring all the other modes of QCD. This is analogous to the quantization of rotational modes of a diatomic molecule, ignoring the vibrational modes. We are guided by two basic principles of effective field theory: (i) The chiral Lagrangian must exhibit the same approximate chiral symmetry as QCD, which means that it must be invariant under $\Sigma \to L\Sigma R^\dagger$ for arbitrary $SU(3)_L \times SU(3)_R$ matrices L, R. We will also be able to incorporate symmetry breaking effects by including the matrix M, requiring that the chiral Lagrangian be invariant under the chiral symmetry if M were to transform as in eq. (52). (ii) The other principle is that the effective theory be an expansion of local operators suppressed by powers of a cutoff Λ, which is set by the scale of physics we are ignoring, such as the ρ, $K*$, ω, and η' mesons (with masses $m_\rho = 770$ MeV, $m_{K^*} = 892$ MeV,

$m_\omega = 782$ MeV and $m_{\eta'} = 958$ MeV). In practice, the cutoff seems to be at $\Lambda \simeq 1$ GeV in many processes. Our calculations will involve an expansion in powers of momenta or meson masses divided by Λ. This cutoff is to be compared with $m_{\pi^\pm} = 140$ MeV, $m_{K^+} = 494$ MeV and $m_\eta = 548$ MeV. For purely mesonic processes, meson masses always appear squared, which helps. Nevertheless, one can surmise that chiral perturbation theory will work far better for pions than kaons or the η. This is a reflection of the fact that $SU(2)_L \times SU(2)_R$ is a much better symmetry of QCD than $SU(3)_L \times SU(3)_R$.

The lowest dimension chirally symmetric operator we can write down is

$$\mathcal{L}_0 = \frac{f^2}{4} \mathrm{Tr}\partial\Sigma^\dagger\partial\Sigma = \mathrm{Tr}\partial\boldsymbol{\pi}\partial\boldsymbol{\pi} + \frac{1}{3f^2}\mathrm{Tr}[\partial\boldsymbol{\pi}, \boldsymbol{\pi}]^2 + \ldots \tag{61}$$

Note that the $f^2/4$ prefactor is fixed by requiring that the mesons have canonically normalized kinetic terms. Thus we have an infinite tower of operators involving a single unknown parameter, f. From the above Lagrangian, it would seem that the only way to determine f is by looking at $\pi\pi$ scattering. However there is a better way: by looking at the charged pion decay $\pi \to \mu\nu$. This occurs through the "semi-leptonic" weak interaction eq. (28), namely the operator

$$\frac{1}{\sqrt{2}} G_F V_{ud} (\overline{u}\gamma^\mu(1-\gamma_5)d)(\overline{\mu}\gamma_\mu(1-\gamma_5)\nu_\mu + \mathrm{h.c.} \tag{62}$$

The matrix element of this operator sandwiched between $|\mu\nu\rangle$ and $\langle\pi|$ factorizes, and the leptonic part is perturbative. We are left with the nonperturbative part,

$$\langle 0|\overline{u}\gamma^\mu(1-\gamma_5)d|\pi^-(p)\rangle \equiv i\sqrt{2}\, f_\pi p^\mu \ . \tag{63}$$

The pion decay constant f_π is determined from the charged pion lifetime to be $f_\pi = 92.4 \pm .25$ MeV.

Even though QCD is nonperturbative, we can easily match this charged current operator onto an operator in the chiral Lagrangian. That is because we can write

$$\overline{u}\gamma^\mu(1-\gamma_5)d = 2\left(j^\mu_{L1} + i j^\mu_{L2}\right) \ , \tag{64}$$

where j^μ_{La} are the eight $SU(3)_L$ currents

$$j^\mu_{La} \equiv \overline{q}\gamma^\mu \left(\frac{1-\gamma_5}{2}\right) T_a q \ . \tag{65}$$

To compute these currents in the effective theory is easy, since we know that under $SU(3)_L$ transformations $\Sigma \to L\Sigma$, or $\delta_{La}\Sigma = iT_a\Sigma$, and can just compute the left-handed currents from the Lagrangian eq. (61) using Noethers theorem. The result is:

$$j^\mu_{La} = -i\frac{f^2}{2}\mathrm{Tr}T_a\Sigma^\dagger\partial^\mu\Sigma = f\,\mathrm{Tr}T_a\partial^\mu\boldsymbol{\pi} + O(\boldsymbol{\pi}^2) \ . \tag{66}$$

In particular,

$$2\left(j^\mu_{L1} + i j^\mu_{L2}\right) = 2f\,\mathrm{Tr}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \partial^\mu\boldsymbol{\pi} + O(\boldsymbol{\pi}^2) = \sqrt{2} f \partial^\mu \pi^- + O(\pi^2) \ , \tag{67}$$

were I made use of eq. (59). Comparing this equation with eq. (63) we see that to this order,

$$f = f_\pi = 93 \text{ MeV} . \tag{68}$$

In general it is not possible to exactly match quark operators with operators in the chiral Lagrangian; it was possible for the semileptonic decays simply because the weak operator factorized into a leptonic matrix element and a hadronic matrix element of an $SU(3)_L$ symmetry current. For a purely hadronic weak decay, such as $K \to \pi\pi$ the four quark operator cannot be factorized, and matching to operators in the chiral Lagrangian involves coefficients which can only be computed on a lattice. Even for these processes the chiral Lagrangian can be predictive, relating weak decays with different numbers of mesons in the final state.

3.3.2 Explicit symmetry breaking

Up to now, I have only discussed operators in the chiral Lagrangian which are invariant. Note that that there are no chirally invariant operators which do *not* have derivatives (other than the operator 1). For example, one cannot write down a chirally invariant mass term for the pions. Recall that without explicit chiral symmetry breaking in the QCD Lagrangian, there would be an infinite number of inequivalent degenerate vacua corresponding to different constant values of the matrix Σ; therefore the energy (and the Lagrangian) can only have operators which vanish when Σ is constant, up to an overall vacuum energy independent of Σ. In fact, rotating $\Sigma \to \Sigma' = \Sigma + i d\theta_a T_a \Sigma$ is an exact symmetry of the theory ($SU(3)_L$), and corresponds to *shifting* the pion fields $\pi_a \to \pi_a + d\theta_a f/2 + O(\pi^2)$. Derivative interactions are a result of this shift symmetry. (In the literature, this is called a *nonlinearly realized* symmetry, which is to say, a spontaneously broken symmetry). A theory of massless particles with nontrivial interactions at zero momentum transfer (such as QCD) would suffer severe infrared divergences, and so if the interactions had not been purely derivative, the theory would either not make sense, or would become nonperturbative like QCD.

This all changes when explicit chiral symmetry breaking is included. Now not all vacua are equivalent, the massless Goldstone bosons become massive "pseudo-Goldstone bosons" (PGBs), and acquire nonderivative interactions. In pure QCD, the only sources of explicit chiral symmetry breaking are instantons (which explicitly break the $U(1)_A$ symmetry, and the quark mass matrix. Electromagnetic interactions also introduce chiral symmetry breaking, as do weak interactions.

To include the effect of quark masses, we need to include the mass matrix M, recalling that if it transformed as in eq. (52), then the theory would have to be invariant. Just as with derivatives, each power of M will be accompanied by $1/\Lambda$. The leading operator we can write down is

$$\mathcal{L}_M = \Lambda^2 f^2 \left(\frac{c}{2} \frac{1}{\Lambda} \mathrm{Tr} M\Sigma + \text{h.c.} \right) \equiv \frac{1}{2} f^2 \mathrm{Tr}(\widetilde{\Lambda} M)\Sigma + \text{h.c.} , \tag{69}$$

where c is an unknown dimensionless coefficient, and I defined

$$c\Lambda \equiv \widetilde{\Lambda} = O(\Lambda) . \tag{70}$$

Expanding to second order in the $\boldsymbol{\pi}$, I get

$$\mathcal{L}_M = -m_\pi^2 \pi^+\pi^- - m_{K^+}^2 K^+K^- - m_{K^0}^2 K^0\overline{K}^0 - \frac{1}{2}\begin{pmatrix}\pi^0 & \eta\end{pmatrix} M_0^2 \begin{pmatrix}\pi^0 \\ \eta\end{pmatrix} \ , \tag{71}$$

with

$$m_\pi^2 = \widetilde{\Lambda}(m_u + m_d) \ , \qquad m_{K^+}^2 = \widetilde{\Lambda}(m_u + m_s) \ , \qquad m_{K^0}^2 = \widetilde{\Lambda}(m_d + m_s) \ , \tag{72}$$

and

$$M_0^2 = \widetilde{\Lambda}\begin{pmatrix}(m_u + m_d) & (m_u - m_d) \\ (m_u - m_d) & \frac{1}{3}(m_u + m_d + 4m_s)\end{pmatrix} \tag{73}$$

Note that (i) the squares of the meson masses are proportional to quark masses; (ii) $\pi^0 - \eta$ mixing is isospin breaking and proportional to $(m_u - m_d)$; (iii) expanding in powers of $(m_u - m_d)$, m_η^2 and $m_{\pi^0}^2$ are given by the diagonal entries of M_0^2, up to corrections of $O\left((m_u - m_d)^2\right)$; (iv) we cannot directly relate quark and meson masses because of the unknown coefficient $\widetilde{\Lambda}$.

Ignoring isospin breaking, the masses obey the Gell-Mann Okuba formula

$$3m_\eta^2 + m_\pi^2 = 4m_K^2 \ . \tag{74}$$

The two sides of the above equation are satisfied experimentally to better than 1% accuracy.

To leading order in α, electromagnetic corrections to the meson masses take the form

$$m_{\pi^+}^2 = \widetilde{\Lambda}(m_u + m_d) + \frac{\alpha}{4\pi}\Delta^2 \ , \qquad m_{K^+}^2 = \widetilde{\Lambda}(m_u + m_s) + \frac{\alpha}{4\pi}\Delta^2 \ , \tag{75}$$

with the neutral particle masses unchanged. In the above formula, Δ has mass dimension 1, and is $O(\Lambda)$ in size; the prefactor of $\alpha/4\pi$ arises since the splitting must arise from a loop diagram involving a photon. Following Weinberg, we can also use the above formula to calculate the ratios of quark masses via the formulas

$$\frac{(m_{K^+}^2 - m_{K^0}^2) - (m_{\pi^+}^2 - m_{\pi^0}^2)}{m_{\pi^0}^2} = \frac{m_u - m_d}{m_u + m_d} \ , \qquad \frac{3m_\eta^2 - m_{\pi^0}^2}{m_{\pi^0}^2} = \frac{4m_s}{m_u + m_d} \ . \tag{76}$$

Plugging in the measured meson masses, the result is

$$\frac{m_u}{m_d} \simeq \frac{1}{2} \ , \qquad \frac{m_d}{m_s} \simeq \frac{1}{20} \ . \tag{77}$$

To specify the quark masses themselves, one must perform a lattice QCD calculation and designate a renormalization scheme. Lattice simulations typically find m_s renormalized at $m = 2$ GeV in the $\overline{MS}$ scheme lies in the $80 - 100$ MeV range, from which one infers from the above ratios $m_d \sim 5$ MeV, $m_d \sim 2.5$ MeV in the same scheme. Evidently most of the mass of baryons and vector mesons does *not* come from the intrinsic masses of the quarks.

3.4 Loops and power counting

What makes the chiral Lagrangian and EFT and not just another model of the strong interactions is that it consists of all local operators consisted with the symmetries of QCD, and that there exists a power counting scheme that allows one to work to a given order, and to be able to make a reliable estimate of the errors arising from neglecting the subsequent order. As discussed in the second lecture, the power counting scheme is intimately related to how one computes radiate corrections in the theory.

Beyond the leading term is an infinite number of chirally invariant operators one can write down which are higher powers in derivatives, as well as operators with more insertions of the quark mass matrix M. The derivative expansion is in powers of ∂/Λ. This power counting is consistent with the leading operator eq. (61), if you consider the chiral Lagrangian to have a prefactor of $\Lambda^2 f^2$, then even in the leading operator derivatives enter as ∂/Λ. Since we have found that meson octet masses scale as $m_\pi^2 \simeq (\widetilde{\Lambda} M)$, and since for on-shell pions $p^2 \sim m^2$, it follows that one insertion of the quark mass matrix is equivalent to two derivatives in the effective field theory expansion. This leads us to write the chiral Lagrangian as a function of (∂/Λ) and $\widetilde{\Lambda} M/\Lambda^2$. Including electromagnetism is straightforward as well: since a derivative $\partial\Sigma$ becomes a covariant derivative $D_\mu\Sigma = \partial_\mu\Sigma - ieA_\mu\,[Q,\Sigma]$, the photon field enters as eA_μ/Λ. Operators arising from electromagnetic loops involve two insertions of the quark charge matrix Q in the proper way (see problem (III.6), along with a loop factor $\alpha/(4\pi)$. Therefore the chiral Lagrangian takes the form

$$\mathcal{L} = \Lambda^2 f^2 \widehat{\mathcal{L}}\left[\Sigma,\, \partial/\Lambda,\, \widetilde{\Lambda} M/\Lambda^2,\, eA/\Lambda,\, (\alpha/4\pi)Q^2\right] \ , \tag{78}$$

where $\widehat{\mathcal{L}}$ is a dimensionless sum of all local, chirally invariant operators (treating M and Q as spurions), where the coefficient of each term (except $\mathcal{L}_0$) is preceded by a dimensionless coefficient to be fit to experiment... which we expect to be $O(1)$, but which may occasionally surprise us! That last assumption is what allows one to estimate the size of higher order corrections.

It should be clear now in what sense the u, d and s are light quarks and can be treated in chiral perturbation theory, while the c, b and t quarks are not: whether the quarks are light or heavy is relative to the scale Λ, namely the mass scale of resonances in QCD. Since the c has a mass ~ 1.5 GeV there is no sensible way to talk about an approximate $SU(4)\times SU(4)$ chiral symmetry and include D, D_s and η_c mesons in our theory of pseudo-Goldstone bosons[4] Of course, you might argue that the strange quark is sort of heavy and should be left out as well, but if we don't live dangerously sometimes, life is too boring.

3.4.1 Subleading order: the $O(p^4)$ chiral Lagrangian

It is a straightforward exercise to write down subleading operators of the chiral Lagrangian. These are operators of $O(p^4)$, $O(p^2M)$ and $O(M^2)$, where M is the quark mass matrix. This was first done by Gasser and Leutwyler, and their choice for the set of operators has

[4] This does not mean that an effective theory for $D-\pi$ interactions is impossible. However, the D mesons must be introduced as heavy matter fields, similar to the way we will introduce baryon fields later, as opposed to approximate Goldstone bosons.

become standard:

$$\begin{aligned}
\mathcal{L}_{p^4} = \; & L_1 \left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right)\right)^2 \\
& + L_2 \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right) \mathrm{Tr}\left(\partial^\mu \Sigma^\dagger \partial^\nu \Sigma\right) \\
& + L_3 \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma \partial_\nu \Sigma^\dagger \partial^\nu \Sigma\right) \\
& + L_4 \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right) \mathrm{Tr}\left(\chi\Sigma + \mathrm{h.c.}\right) \\
& + L_5 \mathrm{Tr}\left(\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right) \left(\chi\Sigma + \mathrm{h.c.}\right)\right) \\
& + L_6 \left(\mathrm{Tr}\left(\chi\Sigma + \mathrm{h.c.}\right)\right)^2 \\
& + L_7 \left(\mathrm{Tr}\left(\chi\Sigma - \mathrm{h.c.}\right)\right)^2 \\
& + L_8 \mathrm{Tr}\left(\chi\Sigma\chi\Sigma + \mathrm{h.c.}\right) \;, \qquad (79)
\end{aligned}$$

where $\chi \equiv \frac{1}{2} f^2 \widetilde{\Lambda}$, where $\widetilde{\Lambda}$ entered in eq. (69). Additional operators involving $F_{\mu\nu}$ need be considered when including electromagnetism.

Note that according to our power counting, we expect the L_i to be of size

$$L_i \sim \frac{\Lambda^2 f^2}{\Lambda^4} = \frac{f^2}{\Lambda^2} \sim 10^{-2} \; . \qquad (80)$$

3.4.2 Calculating loop effects

Now consider loop diagrams in the effective theory. These are often divergent, and so the first issue is how to regulate them. It is easy to show that a momentum cutoff applied naively violates chiral symmetry; and while it is possible to fix that, by far the simplest regularization method is dimensional regularization with a mass independent subtraction scheme, such as $\overline{MS}$.

The $\overline{MS}$ scheme introduces a renormalization scale μ, usually chosen to be $\mu = \Lambda$. However, unlike with cutoff regularization, one never gets powers of the renormalization scale μ when computing a diagram; μ can only appear in logarithms. Consider, for example, the $O(\pi^4)$ operator from $\mathcal{L}_0$, of the form $\frac{1}{f^2}(\partial\pi)^2\pi^2$, and contract the two pions in $(\partial\pi)^2$; this one-loop graph will renormalize the pion mass. However, since the diagram is proportional to $1/f^2$, and no powers of the renormalization scale μ can appear, dimensional analysis implies that any shift in the pion mass from this graph must be proportional to $\delta m_\pi^2 \sim (m_\pi^0)^4/(4\pi f)^2$, times a possible factor of $\ln(m_\pi/\mu)$, where m_π^0 is the mass of the meson at leading order. Here I have included the factor of $1/(4\pi)^2$ that typically arises from a loop diagram. Ignoring the logarithm, compare with this contribution to the pion mass contribution from the $O(p^4)$ chiral Lagrangian, which yields $\delta m_\pi^2 \sim (\mu M)^2/\Lambda^2$. We see that so long as

$$4\pi f_\pi \gtrsim \Lambda \; , \qquad (81)$$

then the contribution from the radiative correction from the lowest order operator is comparable to or smaller than the second order tree-level contribution, up to $\ln m_\pi^2/\mu^2$ corrections.

What about the logarithm? Note that $\ln(\Lambda^2/m_\pi^2) \simeq 4$ for $\mu = 1$ GeV. Therefore a term with a logarithm is somewhat enhanced relative to the higher order tree-level contributions.

It is therefore common to see in the literature a power counting scheme of the form

$$p^2 > p^4 \ln \mu^2/p^2 > p^4 \ldots \tag{82}$$

which means that in order of importance, one computes processes in the following order:

1. Tree level contributions from the $O(p^2)$ chiral Lagrangian;
2. Radiative corrections to the $O(p^2)$ chiral Lagrangian, keeping only $O(p^4 \ln p^2)$ terms;
3. Tree level terms from the $O(p^4)$ chiral Lagrangian, as well as $O(p^4)$ radiative contributions from the $O(p^2)$ chiral Lagrangian;

and so forth. Keeping the logs and throwing out the analytic terms in step #2 is equivalent to saying that most of the $O(p^4)$ chiral Lagrangian renormalized at $\mu = m_\pi$ would come from running induced by the $O(p^2)$ Lagrangian in scaling down from $\mu = \Lambda$ to $\mu = m_\pi$, and not from the initial values of couplings in the $O(p^4)$ Lagrangian renormalized at $\mu = \Lambda$. This procedure would *not* be reasonable in the large N_c limit (see problem (III.7)) but seems to work reasonably well in the real world.

3.4.3 Renormalization of $\langle 0|\bar{q}q|0\rangle$

As an example of a simple calculation, consider the computation of the ratios of the quark condensates,

$$x = \frac{\langle 0|\bar{u}u|0\rangle}{\langle 0|\bar{s}s|0\rangle} \ . \tag{83}$$

Since the operator $\bar{q}q$ gets multiplicatively renormalized, $\langle 0|\bar{q}_i q_i|0\rangle$ is scheme dependent, but the ratio x is not. The QCD Hamiltonian density is given by $\mathcal{H} = \ldots + \bar{q}Mq + \ldots$, and so it follows from the Feynman-Hellman theorem[5] that

$$\langle 0|\bar{q}_i q_i|0\rangle = \frac{\partial}{\partial m_i}\langle 0|\mathcal{H}|0\rangle = \frac{\partial \mathcal{E}_0}{\partial m_i} \ , \tag{84}$$

where $\mathcal{E}_0$ is the vacuum energy density. We do not know what is $\mathcal{E}_0$, but we do know its dependence one the quark mass matrix; from eq. (71)

$$\mathcal{E}_0 = \text{const.} - \frac{1}{2} f^2 \mathrm{Tr}(\widetilde{\Lambda} M)\Sigma + \text{h.c.} + O(M^2 \ln M)\Big|_{\Sigma_{ij}=\delta_{ij}} = f^2 \widetilde{\Lambda} \mathrm{Tr} M + \ldots \ , \tag{85}$$

from which it follows that this scheme

$$\langle 0|\bar{q}_i q_i|0\rangle = \widetilde{\Lambda} f^2 \ , \tag{86}$$

and that in any scheme the leading contributions to x is

$$x = 1 \ . \tag{87}$$

[5]The substance of the Feynman-Hellman theorem is that in first order perturbation theory, the wave function doesn't change while the energy does.

Well good — this is what we started with for massless QCD in eq. (53)! To get the subleading logarithmic corrections, we need to compute the $O(m^2 \ln m^2)$ one-loop correction to the vacuum energy. This loop with no vertices's is the Feynman diagram for which Feynman rules don't work! As easily seen in a Euclidean path integral, the vacuum energy density in a box of 4-volume VT for a real, noninteracting scalar is just

$$\mathcal{E}_0 = -\frac{1}{VT}\ln\left(\det(-\Box + m^2)\right)^{-1/2} = \frac{1}{VT}\frac{1}{2}\mathrm{Tr}\ln(-\Box + m^2) \ . \tag{88}$$

In $d = (4 - 2\epsilon)$ Euclidean dimensions this just involves evaluating for each mass eigenstate the integral

$$\frac{\mu^{4-d}}{2}\int \frac{d^d k}{(2\pi)^d}\ln(k^2 + m^2) \ . \tag{89}$$

where the prefactor of μ^{4-d} was included to keep the mass dimension to equal 4.

Let us first perform the differentiation with respect to quark mass. Then in this scheme we get the correction

$$\delta \langle 0|\bar{q}_i q_i|0\rangle = \frac{1}{2}\sum_a \frac{\partial m_a^2}{\partial m_i}\mu^{4-d}\int \frac{d^d k}{(2\pi)^d}\frac{1}{k^2+m^2} \xrightarrow[\overline{MS}]{} -\sum_a \frac{\partial m_a^2}{\partial m_i}\left(\frac{m_a^2 \ln m_a^2/\mu^2}{32\pi^2}\right) \ , \tag{90}$$

where a is summed over the meson mass eigenstates, and m_i is the mass of the i^{th} flavor of quark. The final result was arrived at after performing the $\overline{MS}$ subtraction (where you only keep the $\ln m^2$ term in the $\epsilon \to 0$ limit; see the appendix B for dimensional regularization formulas).

To the order we are working, the quark condensate ratios are therefore given by

$$\frac{\langle 0|\bar{q}_i q_i|0\rangle}{\langle 0|\bar{q}_j q_j|0\rangle} = 1 - \frac{1}{32\pi^2\widetilde{\Lambda} f^2}\sum_a m_a^2 \ln m_a^2/\mu^2\left(\frac{\partial m_a^2}{\partial m_i} - \frac{\partial m_a^2}{\partial m_j}\right) \ . \tag{91}$$

Using the masses given in eq. (72) and eq. (73), ignoring $\pi^0 - \eta$ mixing, we find

$$x = \frac{\langle 0|\bar{u}u|0\rangle}{\langle 0|\bar{s}s|0\rangle} = 1 - 3g_\pi + 2g_{K^0} + g_\eta + O(m^4) \ , \tag{92}$$

where

$$g_P \equiv \frac{1}{32\pi^2 f^2}m_P^2 \ln\left(\frac{m_P^2}{\mu^2}\right) \tag{93}$$

with $P = \pi$, K^+, K^0, η. The answer is μ dependent, since I have neglected to include the $O(p^4)$ Lagrangian contributions at tree-level, and in fact it is precisely those operators that serve as counterterms for the $1/\epsilon$ poles subtracted in $\overline{MS}$. However, in the usual practice of chiral perturbation theory, I have assumed that with $\mu = \Lambda$, the contributions from the $O(p^4)$ Lagrangian are small compared to the chiral logs I have included. Plugging in numbers with $\mu = 1$ GeV I find

$$g_\pi \simeq -0.028 \ , \quad g_K \simeq -0.13 \ , \quad g_\eta \simeq -0.13 \tag{94}$$

implying that $x \simeq 0.70$ — a 30% correction from the leading result $x = 1$. This is typical of any chiral correction that involves the strange quark, since $m_K^2/\Lambda^2 \simeq 25\%$. Corrections to $\langle \bar{u}u\rangle/\langle \bar{d}d\rangle$ will be *much* smaller, since they depend on isospin breaking, of which a typical measure is $(m_{K^0}^2 - m_{K^+}^2)/\Lambda^2 \simeq 0.004$.

3.5 Including baryons

3.5.1 Transformation properties and couplings

It is interesting to include the baryon octet into the mix. This is reasonable so long as we consider processes with momentum transfer $\ll \Lambda$. Thus we might consider the weak decay $\Lambda \to N\pi$, but not the annihilation $N\overline{N} \to \pi\pi$. There are two separate issues here: (i) How do we figure out how baryons transform under the chiral $SU(3) \times SU(3)$ symmetry so that we can couple them to Σ, and (ii) do we need or desire the Dirac spinor formulation if we are only going to consider low momentum transfer processes? I will address the first question first, using Dirac spinors. Then I will introduce the heavy baryon formalism of Jenkins and Manohar, replacing the Dirac spinors.

First consider a world where the u, d and s are massless. We know that the baryons transform as an octet under the unbroken $SU(3)_V$ symmetry, but how do they transform under $SU(3) \times SU(3)$? The answer is: just about any way you want. To see this, consider a baryon field B written as a 3×3 traceless matrix of Dirac spinors, transforming as an octet under $SU(3)_V$:

$$B \to VBV^\dagger \ . \tag{95}$$

By considering T_3 and T_8 transformations it is possible to determine the form of the matrix B, just as we did for the meson matrix $\boldsymbol{\pi}$:

$$B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix} \ . \tag{96}$$

Now construct the left- and right-handed baryons $B_{R,L} = \frac{1}{2}(1 \pm \gamma_5)B$. Suppose that B transformed as the $(8,1) \oplus (1,8)$ representation under $SU(3) \times SU(3)$, namely

$$B_R \to RB_RR^\dagger \ , \qquad B_L \to LB_LL^\dagger \ . \tag{97}$$

But then I could define

$$B'_R \equiv \Sigma B_R \ , \qquad B'_L = B_L\Sigma \ . \tag{98}$$

The new field B' works equally well as a baryon field. However B' transforms as a $(3, \overline{3})$ under $SU(3) \times SU(3)$:

$$B' \to LB'R^\dagger \ . \tag{99}$$

Note that both B and B' both transform properly as an $SU(3)_V$ octet when $R = L \equiv V$, as in eq. (95). What we are seeing is that when you have massless pions around, you can't tell the difference between a baryon, and a superposition of that baryon with a bunch of zero momentum massless pions, and yet the two will have different $SU(3) \times SU(3)$ transformation properties. This is not a problem — rather it is liberating. It means we can choose whatever $SU(3) \times SU(3)$ transformation rule we wish for the baryons, so long as eq. (95) still holds.

While the basis eq. (97) looks appealing, it has its drawbacks. For example it allows the interaction

$$M \mathrm{Tr} \overline{B}_L \Sigma B_R + \text{h.c.} = M \mathrm{Tr} \overline{B} B + \frac{2M}{f} \mathrm{Tr} \overline{B} i \gamma_5 \boldsymbol{\pi} B + \ldots \tag{100}$$

which makes it *look* like the pion can have nonderivative couplings...which ought to be impossible for a Goldstone boson. However, the γ_5 coupling is in fact a derivative coupling at low momentum transfer, made obscure.

A better basis is the following. Define

$$\xi = e^{i\boldsymbol{\pi}/f} = \sqrt{\Sigma} \ . \tag{101}$$

One can show that under an $SU(3) \times SU(3)$ transformation

$$\xi \to L \xi U^\dagger = U \xi R^\dagger \ , \tag{102}$$

where U is a uniquely defined matrix which is in general a function of the constant L and R matrices characterizing the $SU(3) \times SU(3)$ transformation, as well as the $\boldsymbol{\pi}(x)$ field. For the special case of $SU(3)_V$ transformations, $R = L = V$ and it is easy to show that $U = V$ as well, independent of $\boldsymbol{\pi}$. For more general $SU(3) \times SU(3)$ transformations U is a mess, but we will not need to know its exact form. Now if we take the basis eq. (97) and replace $B_L \to \xi^\dagger B_L \xi$, $B_R \to \xi B_R \xi^\dagger$, we get a new basis where left- and right-handed components of B transform the same way, namely

$$B \to U B U^\dagger \ . \tag{103}$$

Given the above transformation rule, B cannot couple directly to Σ, but we can define the axial and vector currents and chiral covariant derivative:

$$\begin{aligned}
A_\mu &\equiv \frac{i}{2}\left(\xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger\right) \xrightarrow[SU(3)\times SU(3)]{} U A_\mu U^\dagger \ , \\
V_\mu &= \frac{1}{2}\left(\xi^\dagger \partial_\mu \xi + \xi \partial_\mu \xi^\dagger\right) \xrightarrow[SU(3)\times SU(3)]{} U V_\mu U^\dagger + U \partial_\mu U^\dagger \ , \\
D_\mu B &\equiv \left(\partial_\mu B + [V_\mu, B]\right) \xrightarrow[SU(3)\times SU(3)]{} U (D_\mu B) U^\dagger \ .
\end{aligned} \tag{104}$$

Armed with this formalism we can write down an effective theory for meson-baryon interactions, whose first few chirally symmetric terms are

$$\mathcal{L}_1 = \mathrm{Tr} \overline{B} (\gamma^\mu D_\mu - m_0) B - D \mathrm{Tr} \overline{B} \gamma^\mu \gamma_5 \{A_\mu, B\} - F \mathrm{Tr} \overline{B} \gamma^\mu \gamma_5 [A_\mu, B] \ . \tag{105}$$

Note that the common octet mass m_0 is chirally symmetric, independent of the quark masses.

Expanding A_μ and V_μ in the meson fields,

$$A_\mu = -\frac{1}{f}\partial_\mu \boldsymbol{\pi} + O(\boldsymbol{\pi}^3) \ , \qquad V_\mu = \frac{1}{2f^2}\left(\boldsymbol{\pi}\partial_\mu \boldsymbol{\pi} - (\partial_\mu \boldsymbol{\pi})\boldsymbol{\pi}\right) + O(\boldsymbol{\pi}^4) \ . \tag{106}$$

For non-relativistic baryons, the Dirac analysis implies that $\overline{B}\gamma^0 B$ and $\overline{B}\vec{\gamma}\gamma_5 B$ are big, equal to 1 and $\vec{S}$ (the baryon spin) respectively; in contrast, the bilinears $\overline{B}\vec{\gamma}B$ and $\overline{B}\gamma^0\gamma_5 B$ are small, given by $\vec{q}/m_0$ and $(\vec{S}\cdot\vec{q})/m_0$ respectively, where $\vec{q}$ is the 3-momentum transfer. Therefore the leading meson-baryon interactions for nonrelativistic baryons (written as 2-component spinors) is

$$\frac{1}{2f^2}\,\mathrm{Tr}\left(B^\dagger[\boldsymbol{\pi}\dot{\boldsymbol{\pi}}-\dot{\boldsymbol{\pi}}\boldsymbol{\pi},B]\right)+D\mathrm{Tr}\left(B^\dagger\vec{\sigma}\cdot\{\vec{\nabla}\boldsymbol{\pi},B\}\right)+F\mathrm{Tr}\left(B^\dagger\vec{\sigma}\cdot\left[\vec{\nabla}\boldsymbol{\pi},B\right]\right)\ . \tag{107}$$

The vector current interaction is also called the Weinberg-Tomazawa term; it does not involve any unknown parameters and is required by chiral symmetry. The axial current interaction involves two new couplings D and F that may be fit to semileptonic baryon decay, using the same wonderful coincidence exploited in relating f to f_π: namely that weak charged currents happen to be $SU(3)$ currents, and so can be unambiguously computed in the effective theory. The combination $(D+F)=g_A=1.25$ is derived from neutron decay. From hyperon decay, one determines $F\simeq 0.44$, $D\simeq 0.81$. Because the axial interactions involve derivatives of the mesons, they contribute to p-wave scattering, but not s-wave; in contrast, the vector interaction contributes to s-wave scattering.

The above Lagrangian is $O(p)$, involving single derivatives on the meson fields. Therefore symmetry breaking terms involving the quark mass matrix M are subleading, as we have seen that $M\sim p^2$ in our power counting. There are three such terms, are

$$\mathcal{L}_2 \;=\; a_1\mathrm{Tr}\overline{B}(\xi^\dagger M\xi+\mathrm{h.c.})B+a_2\mathrm{Tr}\overline{B}B(\xi^\dagger M\xi+\mathrm{h.c.})+a_3\mathrm{Tr}\left(M\Sigma+\mathrm{h.c.}\right)\mathrm{Tr}\overline{B}B \tag{108}$$

A combination of three of the mass parameters m_0, a_1, a_2, and a_3 may be determined from the octet masses; to fit the fourth combination requires additional data, but can be done (see problem (III.9)). At the same order one should consider chirally symmetric terms with two derivatives acting on the mesons, such as subleading terms in the nonrelativistic expansion of $\mathcal{L}_1$, as well as new terms such as $\mathrm{Tr}\overline{B}A_\mu A^\mu B$. However, I believe that there are too many such subleading terms to fit to low energy meson-baryon scattering data.

3.5.2 The heavy baryon formulation

It was clearly awkward to have to start with Dirac spinors, and then take the nonrelativistic limit on order to disentangle the order of each contribution in the chiral expansion. Furthermore, the EFT expansion for meson-baryon interactions is complicated by the fact that for on-shell baryons, $i\partial_t B\sim m_0 B$, and $m_0\sim\Lambda$. So how can there be a derivative expansion since ∂_t/Λ and m_0/Λ are both $O(1)$? The answer is that for small momentum transfer, all one ever sees is the combination $(i\partial_t-m_0)$, which is small. However, a cancellation between two large parameters is the bane of effective field theory...it makes power counting obscure.

Another issue is that there is the Δ resonance not too far above threshold in πN scattering.

Both complications were addressed in the Jenkins-Manohar approach [9,10], which applied the formalism developed for HQET, which I discussed briefly in my first lecture. One defines the baryon field B_{v} for baryons with velocity v_μ (a 3×3 traceless matrix)

$$B_{\mathrm{v}}(x)=e^{im_0\not{v}_\mu x^\mu}B(x)\ , \tag{109}$$

where B is the baryon field I introduced above. They also introduce spin operators $S_{\rm v}^\mu$ satisfying

$$v_\mu S_{\rm v}^\mu = 0 \ , \quad S_{\rm v}^2 B_{\rm v} = -\frac{3}{4} B_{\rm v} \ , \quad \{S_{\rm v}^\alpha, S_{\rm v}^\beta\} = \frac{1}{2}(v^\alpha v^\beta - \eta^{\alpha\beta}) \ , \quad \left[S_{\rm v}^\alpha, S_{\rm v}^\beta\right] = i\epsilon^{\alpha\beta\mu\nu} v_\mu S_{v\nu} \tag{110}$$

In terms of these operators one can work out the Dirac bilinears such as

$$\overline{B}_{\rm v}\gamma^\mu B_{\rm v} = v^\mu \overline{B}_{\rm v} B_{\rm v} \ , \qquad \overline{B}_{\rm v}\gamma^\mu\gamma_5 B_{\rm v} = 2\overline{B}_{\rm v} S_{\rm v}^\mu B_{\rm v} \ , \tag{111}$$

etc. It is then possible to rewrite the leading terms in the baryon Lagrangian of eq. (110) as

$$\mathcal{L}_1 = i\mathrm{Tr}\overline{B}_{\rm v}(v_\mu D^\mu) B_{\rm v} + 2D\mathrm{Tr}\overline{B}_{\rm v} S_{\rm v}^\mu \{A_\mu, B_{\rm v}\} + 2F\mathrm{Tr}\overline{B}_{\rm v} S_{\rm v}^\mu \left[A_\mu, B_{\rm v}\right] \ . \tag{112}$$

Note that the large baryon mass m_0 has disappeared from the expression.

The decuplet field $T_{\rm v}^\mu$ consisting of the Δ, Σ^*, Ξ^* and Ω may be introduced in a similar way, with a few additional complications coming from the fact that they possess spin $\frac{3}{2}$, and therefore have to be represented as Rarita-Schwinger fields instead of Dirac spinors, and because, as a decuplet of $SU(3)_V$, they form a 3-index symmetric tensor, rather than a 3×3 matrix like the baryon octet.

3.6 Problems

III.1) Verify eq. (59).

III.2) How does Σ transform under P (parity)? What does this transformation imply for the intrinsic parity of the π_a mesons? How does Σ transform under C (charge conjugation)? Which of the mesons are eigenstates of CP, and are they CP even or odd? Recall that under P and C the quarks transform as

$$\begin{aligned} P: &\quad q \to \gamma^0 q \ , \\ C: &\quad q \to C\bar{q}^T \ , \qquad C = C^\dagger = C^{-1} = -C^T \ , \quad C\gamma_\mu C = -\gamma_\mu^T \ , \qquad C\gamma_5 C = \gamma_5 \ . \end{aligned} \tag{113}$$

III.3) How do we know that c, and hence $\widetilde{\Lambda}$, is positive in eq. (69)? How would the world look different if it were negative? Hint: consider what Σ matrix would minimize the vacuum energy, and its implications for the spectrum of the theory.

III.4) An axion is a hypothetical particle proposed to explain why the electric dipole moment of the neutron is so small (the strong CP problem). It couples to quarks through the quark mass matrix, where one makes the substitution

$$M \to M e^{iaX/f_a} \tag{114}$$

in eq. (51), where a is the axion field, f_a is the axion decay constant, and X is a 3×3 diagonal matrix constrained to have $\mathrm{Tr}X = 1$. Compute the axion mass in terms of m_π, f_π and f_a, dropping terms of size $m_{u,d}/m_s$. Hint: use the remaining freedom in choosing X to ensure that the axion does not mix with the π^0 or the η mesons.

III.5) Compute the current left-handed current j^μ_{La} through $O(\pi^3)$. Draw the one-loop diagrams that alter the relation between f and f_π. Estimate the contribution proportional to $\ln(m_\pi^2/\Lambda^2)$

III.6) Include electromagnetism in the QCD Lagrangian eq. (50) by having the photon couple to left- and right-handed quarks through the charge matrix Q_L and Q_R respectively. What are these two matrices? How would they have to transform under $SU(3)_L \times SU(3)_R$ if the theory was to remain chirally invariant? Treating $Q_{L,R}$ as spurions, show how to modify the leading term in the chiral Lagrangian to ensure that it is gauge invariant under $U(1)_{\rm em}$? Write down the leading nonderivative operator involving powers of Σ, $\Sigma^\dagger$, and a pair of Q's. Show that this operator contributes to the masses of the charged pseudoscalars as given in eq. (75). Does Δ^2 in eq. (75) arise from a contact interaction, or a photon loop on a meson line? (Hint: think about renormalization schemes.) Approximately how big would the contribution to Δ be from a 1-loop diagram with momentum cutoff Λ?

III.7) Is the relation eq. (81) obeyed in QCD for large N_c (where N_c is the number of colors, and $N_c = 3$ in the real world)?

III.8) Draw the Feynman diagrams that lead to $B\pi \to B\pi$ scattering at leading order in chiral perturbation theory.

III.9) Assuming no isospin breaking ($m_u = m_d = \overline{m}$), compute the baryon octet masses in terms of m_0, a_1, a_2 and a_3. Why can't one determine these four parameters from the four masses M_N, M_Σ, M_Λ and M_Ξ? The failure to determine four parameters from four masses implies that there must be a relationship predicted for the baryon masses — the Gell-Mann Okuba formula for baryons. What is it? What sort of additional data would one need to uniquely determine all four constants?

4 Effective theories of nucleons

This is a nuclear physics school, and in the last lecture we finally mentioned nucleons! So the obvious question is: can the effective field theory for nucleons cast any light on the properties of nuclei or nuclear matter? Today I will tell you about a couple of approaches in that direction. The first uses the chiral Lagrangian derived yesterday in the mean field approximation, and shows how Bose-Einstein condensation of pions or kaons might occur. The second goes beyond mean field theory, and applies effective field theory to low energy few-body scattering. I will mention some issues in trying to push these technologies farther. An amusing spinoff is that the techniques we develop can be profitably used to analyze systems of trapped atoms.

4.1 Kaon condensation

In the early seventies it was suggested by A. B. Migdal that pions might Bose condense in nuclear matter. His reason for thinking this is that pions have attractive p-wave interactions with nucleons (although repulsive s-wave interactions). A decade of research showed that competition between s-wave repulsion and p-wave attraction made pion condensation dicey at best. About 15 years later Ann Nelson and I proposed that kaon condensation (K^- condensation, to be precise) might be favored instead [11]. This seemed surprising, given that the binding energy per nucleon in iron is about 9 MeV, while the binding energy for a K^- in nuclear matter would have to exceed the kaon mass (490 MeV in vacuum) in order to lead to Bose condensation! Another couple of decades work, and the consensus is that a binding energy of several$\times$100 MeV is not unreasonable for a K^- in dense matter, but the details make it unclear whether kaon condensation could occur at $3\times$ nuclear density as we predicted, or at much higher densities, or never as other forms of strange quark matter or hyperon matter appear instead.

The mean field calculation for meson condensation is rather easy to understand: (i) one assumes a background meson field; (ii) one computes the dispersion relation for the baryon octet in this background meson field, in the presence of chemical potentials for baryon number and electric charge; (iii) one occupies all the fermion energy levels up to the Fermi energy, for the baryon octet and for electrons; (iv) one minimizes the total free energy of the system as a function of the background meson field, subject the constraint of charge neutrality. This approach ignores a host of important effects, such as spatial correlations of particles and other nuclear forces beside that of octet meson exchange.

While conceptually simple, a less ambitious and calculationally more simple problem to solve is to compute the critical density for which meson condensation appears. For this one need only expand the free energy to second order in the meson field, and look for an instability — that is, a negative mass squared for the meson. This calculation was performed by Politzer and Wise in ref. [12], in greater generality than I will do here.

Consider a system with neutrons, protons, electrons and a spatially homogeneous kaon condensate, with a chemical potential for electrons, μ_e, to constrain the system to be charge neutral. The kaon condensate corresponds to a classical field

$$K^- = v e^{-i\mu t} \ , \tag{115}$$

where the kaon chemical potential $\mu = \mu_e$ equals the electron chemical potential since the K^- also carries electric charge $-e$. The amplitude v is unknown. Now plug this field into the chiral Lagrangian, and calculate the neutron mass to $O(v^2)$, setting $m_u = m_d = 0$. One finds

$$\Delta M_N = (M_N(v) - M_N(0)) = \frac{|v|^2}{2f^2}\left(-\mu_e + (4a_3 + 2a_2)m_s\right) \ , \tag{116}$$

where the negative first term is from the attractive Weinberg-Tomazawa s-wave interaction. It is possible to determine $a_2 m_s$ by calculating the baryon octet mass splittings (problem (III.9)) with the result

$$a_2 m_s = \frac{1}{2}(M_\Sigma - M_N) = 134 \text{ MeV} \ . \tag{117}$$

However, $a_3 m_s$ cannot be determined from the baryon spectrum, but rather from pion-nucleon scattering. The result is very uncertain. The a_3 operator can be related to the dependence of the nucleon mass on the strange quark mass:

$$m_s \frac{\partial M_N}{\partial m_s} = -2(a_2 + a_3)m_s \ . \tag{118}$$

Some extractions have given this quantity to be $+350$ MeV, corresponding to $(2a_3 + a_2)m_s = -450$ MeV. On the other hand, if the strange quark were very heavy, then one could calculate the above quantity in perturbative QCD, with the result $m_s \frac{\partial M_N}{\partial m_s} = (2/29)M_N = +70$ MeV — which would correspond to $(2a_3 + a_2)m_s = -205$ MeV. As for the sixe of μ_e, it is typically ~ 200 MeV in the cores of neutron stars in conventional calculations.

So turning on a kaon vev v lowers the neutron mass, an effect that save more energy density the denser the neutrons. On the other hand it costs energy to make kaons because of their mass. The net energy density is then

$$m_K^2 |v|^2 - \rho\Delta M_N \ = |v|^2\left(m_K^2 - \frac{\rho}{2f^2}\left(\mu_e - (4a_3 + 2a_2)m_s\right)\right) \ , \tag{119}$$

and we see an instability at the critical neutron density

$$\rho_c = f^2 \frac{m_K^2}{\mu_e/2 - (2a_3 + a_2)m_s} \ . \tag{120}$$

For a crude calculation you can take nuclear density to equal $\rho_0 \simeq f^2 m_\pi$, so we get, if we ignore the μ_e term,

$$\frac{\rho_c}{\rho_0} \simeq -\frac{m_K^2}{m_\pi (2a_3 + a_2)m_s} \ , \tag{121}$$

which ranges between 3.5 and 8.5 depending on what we take for eq. (118). However this simple calculation drops a lot of important effects: (i) the chemical potential μ_e will lower ρ_c; (ii) with K^- one can have nearly equal numbers of protons and neutrons and still have charge neutrality, gaining for you the nuclear symmetry energy; (iii) correlations will tend to make condensation less likely than appears in a mean field calculation, and more. I think the best progress on the subject in the near term will come from a lattice calculation of $a_3 m_s$.

4.2 An EFT for pionless nucleon-nucleon scattering

We would like to go beyond mean field theory and at least set up an honest, effective field theory formulation for nucleon-nucleon interactions which could be used to compute properties of nuclear matter in terms of a few phenomenological parameters which could be fit to data. This contrasts with the method widely used of constructing a nucleon-nucleon potential, tweaking it until it is capable of fitting all of the nucleon-nucleon scattering phase shift data, and then using that potential to perform an N-body calculation, adding three-body interactions as needed to explain the data. Both methods make use of phenomenological parameters to fit the data, both methods are predictive in the sense that there is far more data than free parameters. So how would an effective field theory treatment be any better than using a potential model? There are several reasons:

i. Working in an EFT to a given order is equivalent to including all the operators necessary to reproduce QCD to a given accuracy in a p/Λ expansion. In contrast, in a potential model one never knows whether the next observable one encounters will be calculable to the same accuracy with which the model fits previous observables. If the effects of an operator in the EFT have been left out of the potential models, then the different potential models will disagree on observables, but they will fall on a curve the potential models have omitted a free parameter corresponding to a certain operator in the EFT, then they will disagree among themselves, but their predictions will fall upon a curve. This curve represents the arbitrary value the model assigns to a free parameter in the EFT.

ii. Unlike potential models, EFT never suffers from any ambiguity about "on-shell" and "off-shell" interactions.

iii. Relativistic effects, such as time retardation are simple to include in an EFT, but not in a potential model.

iv. Dynamical processes, scattering, inelastic collisions — all of these are more simply treated in an EFT.

For an effective field theory to apply, however, it is necessary that there exist a gap in the spectrum. For momentum transfer far below the pion mass, this is clearly the case: then one can use a pion-less EFT, where the only interactions are n-body nucleon contact interactions (as well as electromagnetism). I will start by describing such a theory, which has received a lot of attention over the past decade.

4.3 The pionless EFT for nucleon-nucleon interactions

Given that the pionless theory consists only of nonrelativistic nucleons, the Lagrangian is quite simple. It takes the forms:

$$\begin{aligned}\mathcal{L}_{eff} &= N^\dagger \left(i\partial_t + \nabla^2/2M\right) N \\ &\quad +(\mu/2)^{4-d}\left[C_0(N^\dagger N)^2 + \frac{C_2}{8}\left[(NN)^\dagger(N\overleftrightarrow{\nabla}^2 N) + h.c\right] + ...\right] ,\end{aligned} \tag{122}$$

where

$$\overleftrightarrow{\nabla}^2 \equiv \overleftarrow{\nabla}^2 - 2\overleftarrow{\nabla}\cdot\overrightarrow{\nabla} + \overrightarrow{\nabla}^2 . \tag{123}$$

I have suppressed everywhere the spin and isospin indices and their contractions which project these interactions onto the various scattering channels ($^1S_0$, *siii*, etc.). The ellipsis indicates higher derivative operators, and $(\mu/2)$ is an arbitrary mass scale introduced to allow the couplings C_{2n} multiplying operators containing ∇^{2n} to have the same dimension for any D. I focus on the s-wave channel (generalization to higher partial waves is straightforward), and assume that M is very large so that relativistic effects can be ignored. The form of the C_2 operator is fixed by Galilean invariance, which implies that when all particle momenta are boosted $\mathbf{p} \to \mathbf{p} + M\mathbf{v}$, the Lagrangian must remain invariant. There exists another two derivative operator for p-wave scattering which I will not be discussing.

The usual effective field theory expansion requires one to identify a class of diagrams to sum which gives the amplitude $i\mathcal{A}$ to the desired order in a p/Λ expansion. For nonrelativistic scattering, the scattering amplitude is related to the S-matrix by

$$S = 1 + i\frac{Mp}{2\pi}\mathcal{A} \ , \tag{124}$$

where $p = \sqrt{ME_{\rm cm}}$ is the magnitude of the momentum that each nucleon has in the center of momentum frame. For s-wave scattering, $\mathcal{A}$ is related to the phase shift δ by

$$\mathcal{A} = \frac{4\pi}{M}\frac{1}{p\cot\delta - ip} \ . \tag{125}$$

However, it is well known that for a short-range two-body potential $V(r)$ that dies off exponentially fast for $r\Lambda > 1$, it is not $\mathcal{A}$ which in general has a good Taylor expansion in p/Λ, but rather the quantity $p\cot\delta$ which is expanded as:

$$p\cot\delta = -\frac{1}{a} + \frac{1}{2}\Lambda^2 \sum_{n=0}^{\infty} r_n \left(\frac{p^2}{\Lambda^2}\right)^{n+1} \ . \tag{126}$$

This is called the effective range expansion, were a is the scattering length, and r_0 is the effective range. At best, our effective theory for nucleon-nucleon scattering should reproduce the effective range expansion. Of course, if this were the whole story, it would be boring, reproducing well known results! What will make it more interesting is when we incorporate electromagnetic and weak interactions into the theory. But first we need to understand the power counting of our EFT, since Feynman diagrams give one the amplitude $\mathcal{A}$ and not the quantity $p\cot\delta$. A Taylor expansion of $\mathcal{A}$ in powers of p yields

$$\mathcal{A} = -\frac{4\pi a}{M}\left[1 - iap + (ar_0/2 - a^2)p^2 + O(p^3/\Lambda^3)\right] \ , \tag{127}$$

For a generic short-range potential, the coefficients r_n are generally $O(1/\Lambda)$ for all n. However, a can take on any value, which is problematic since for $1/|a| > \Lambda$, the above momentum expansion of the amplitude has a radius of convergence set by $1/|a|$ and not by Λ. A general property of the scattering length is that $1/a$ is negative for a weakly attractive potential, vanishes for a more attractive potential which possesses a boundstate at threshold, and becomes positive for an even more attractive potential with a deep boundstate. (For example: if one considers an attractive Yukawa potential for the form

$$V(r) = -\frac{g^2}{4\pi}\frac{e^{-\Lambda r}}{r} \tag{128}$$

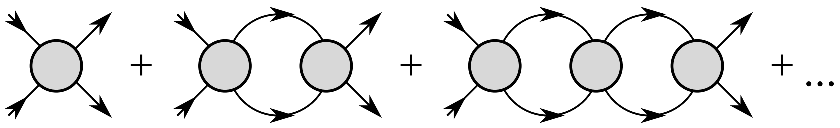

Figure 5: *The bubble chain arising from local operators. The vertex is given by the tree level amplitude, eq. (129).*

then a bound state at threshold appears for the critical coupling $\eta \equiv g^2 M/(4\pi\Lambda) \simeq 1.7$, at which point the scattering length a diverges.) First I consider the situation where the scattering length is of natural size $|a| \sim 1/\Lambda$, and then I discuss the case $|a| \gg 1/\Lambda$, which is relevant for realistic NN scattering.

4.3.1 The case of a "natural" scattering length: $1/|a| \simeq \Lambda$

In the regime $|a| \sim 1/\Lambda$ and $|r_n| \sim 1/\Lambda$, the expansion eq. (127) of the amplitude $\mathcal{A}$ which converges up to momenta $p \sim \Lambda$, and it is this expansion that we wish to reproduce in an effective field theory.

The complete tree level s partial wave amplitude in the center of mass frame arising from $\mathcal{L}_{eff}$ is

$$i\mathcal{A}^{(cm)}_{\rm tree} = -i(\mu/2)^{4-D}\sum_{n=0}^{\infty} C_{2n}(\mu)p^{2n} \ , \tag{129}$$

where the coefficients $C_{2n}(\mu)$ are the couplings in the Lagrangian of operators with $2n$ gradients contributing to s-wave scattering. One may always trade time derivatives for spatial gradients, using the equations of motion when computing S-matrix elements, and so I will ignore such operators.

Beyond tree level one encounters the loop diagrams shown in Fig. 5. Formally, these are *all* the diagrams one encounters in a nonrelativistic theory...if you cut the diagrams in half somewhere in the middle, you can only encounter the two original particles and no additional particle-antiparticle pairs. The loop integrals one encounters are all of the form

$$\begin{aligned}
I_n &\equiv i(\mu/2)^{4-D}\int \frac{\mathrm{d}^D q}{(2\pi)^D}\frac{\mathbf{q}^{2n}}{\left(E/2+q_0-\frac{\mathbf{q}^2}{2M}+i\epsilon\right)\left(E/2-q_0-\frac{\mathbf{q}^2}{2M}+i\epsilon\right)}\\
&= (\mu/2)^{4-D}\int\frac{\mathrm{d}^{(D-1)}\mathbf{q}}{(2\pi)^{(D-1)}}\,\mathbf{q}^{2n}\left(\frac{1}{E-\mathbf{q}^2/M+i\epsilon}\right)\\
&= -M(ME)^n(-ME-i\epsilon)^{(D-3)/2}\Gamma\left(\frac{3-D}{2}\right)\frac{(\mu/2)^{4-D}}{(4\pi)^{(D-1)/2}} \ .
\end{aligned}\tag{130}$$

In order to define the theory, one must specify a subtraction scheme; different subtraction schemes amount to a reshuffling between contributions from the vertices and contributions from the the UV part of the loop integration. How does one choose a subtraction scheme that is useful? I am considering the case $|a|, |r_n| \sim 1/\Lambda$, and wish to reproduce the expansion of the amplitude eq. (127). In order to do this via Feynman diagrams, it is convenient if any Feynman graph with a particular set of operators at the vertices only contributes to the expansion of the amplitude at a particular order. Since the the expansion eq. (127) is a strict Taylor expansion in p, it is it is therefore very convenient if each Feynman graph gives one a simple monomial in p. Obviously, this won't be true in a random subtraction scheme. A subtraction scheme that fulfills this criterion is the minimal subtraction scheme (MS) which amounts to subtracting any $1/(D-4)$ pole before taking the $D \to 4$ limit. As the integral eq. (130) doesn't exhibit any such poles, the result in MS is simply

$$I_n^{MS} = (ME)^n \left(\frac{M}{4\pi}\right) \sqrt{-ME - i\epsilon} = -i \left(\frac{M}{4\pi}\right) p^{2n+1} \ . \tag{131}$$

Note the nice feature of this scheme that the factors of q inside the loop get converted to factors of p, the external momentum. Similarly, a factor of the equations of motion, $i\partial_t + \nabla^2/2M$, acting on one of the internal legs at the vertex, causes the loop integral to vanish. Therefore one can use the on-shell, tree level amplitude eq. (129) as the internal vertices in loop diagrams; summing the bubble diagrams in the center of mass frame gives

$$\mathcal{A} \ = \ -\frac{\sum C_{2n} p^{2n}}{1 + i(Mp/4\pi) \sum C_{2n} p^{2n}} \ . \tag{132}$$

Since there are no poles at $D = 4$ in the MS scheme, the coefficients C_{2n} are independent of the subtraction point μ. The power counting in the MS scheme is particularly simple, as promised:

1. Each propagator counts as $1/p^2$;
2. Each loop integration $\int d^4 q$ counts as p^5 (since $q_0 \sim \mathbf{q}^2/2M$);
3. Each vertex $C_{2n}\nabla^{2n}$ contributes p^{2n}.

The amplitude may be expanded in powers of p as

$$\mathcal{A} = \sum_{n=0}^{\infty} \mathcal{A}_n \ , \qquad \mathcal{A}_n \sim O(p^n) \tag{133}$$

where the $\mathcal{A}_n$ each arise from graphs with $L \leq n$ loops and can be equated to the low energy scattering data eq. (127) in order to fit the C_{2n} couplings. In particular, $\mathcal{A}_0$ arises from the tree graph with C_0 at the vertex; $\mathcal{A}_1$ is given by the 1-loop diagram with two C_0 vertices; $\mathcal{A}_2$ is gets contributions from both the 2-loop diagram with three C_0 vertices, as well as the tree diagram with one C_2 vertex, and so forth. Thus the first three terms are

$$\mathcal{A}_0 = -C_0 \ , \qquad \mathcal{A}_1 = iC_0^2 \frac{Mp}{4\pi} \ , \qquad \mathcal{A}_2 = C_0^3 \left(\frac{Mp}{4\pi}\right)^2 - C_2 p^2 \ . \tag{134}$$

Comparing eqs. (127, 134) I find for the first two couplings of the effective theory

$$C_0 = \frac{4\pi a}{M} \ , \qquad C_2 = C_0 \frac{a r_0}{2} \ . \tag{135}$$

In general, when the scattering length has natural size,

$$C_{2n} \sim \frac{4\pi}{M\Lambda}\frac{1}{\Lambda^{2n}} \ . \tag{136}$$

Note that the effective field theory calculation in this scheme is completely perturbative even though the underlying short-distance physics need not be. Also note that our choice of subtraction scheme (MS), while not changing the physics, made the power counting transparent. A feature of the fact that we are computing consistently to a given order in momentum is that fact that our results are independent of the renormalization scale μ.

4.3.2 The realistic case of an "unnatural" scattering length

One might guess that the results of the previous section would apply to NN scattering, with role of Λ played by m_π or $m_\pi/2$. However, while it is true that the pion is the lightest hadron exchanged between nucleons, the EFT is much more interesting than the above scenario, as the NN scattering lengths are unnaturally large. For example, the ${}^1S_0$ scattering length is $a_0 = -23.714 \pm .013$ fm $\simeq 1/(8$ MeV$)$, which is *much* bigger than $1/m_\pi \simeq 1/(140$ MeV$)$.

For a nonperturbative interaction with a boundstate near threshold, the expansion of $\mathcal{A}$ in powers of p is of little practical value, as it breaks down for momenta $p \gtrsim 1/|a|$, far below Λ. In the above effective theory, this occurs because the couplings C_{2n} are anomalously large, $C_{2n} \sim 4\pi a^{n+1}/M\Lambda^n$. However, the problem is not with the effective field theory method, but rather with the subtraction scheme chosen.

Instead of reproducing the expansion of the amplitude shown in eq. (127), one needs to expand in powers of p/Λ while retaining ap to all orders:

$$\mathcal{A} = -\frac{4\pi}{M}\frac{1}{(1/a+ip)}\left[1+\frac{r_0/2}{(1/a+ip)}p^2+\frac{(r_0/2)^2}{(1/a+ip)^2}p^4+\frac{(r_1/2\Lambda^2)}{(1/a+ip)}p^4+\ldots\right] \tag{137}$$

Note that for $p > 1/|a|$ the terms in this expansion scale as $\{p^{-1}, p^0, p^1, \ldots\}$. Therefore, the expansion in the effective theory should take the form

$$\mathcal{A} = \sum_{n=-1}^{\infty} \mathcal{A}_n \ , \qquad \mathcal{A}_n \sim O(p^n) \tag{138}$$

beginning at $n = -1$ instead of $n = 0$, as in the expansion eq. (133). Comparing with eq. (137), we see that

$$\begin{aligned} \mathcal{A}_{-1} &= -\frac{4\pi}{M}\frac{1}{(1/a+ip)} \ , \\ \mathcal{A}_0 &= -\frac{4\pi}{M}\frac{r_0 p^2/2}{(1/a+ip)^2} \ , \end{aligned} \tag{139}$$

and so forth. Again, the task is to compute the $\mathcal{A}_n$ in the effective theory, and equate to the appropriate expression above, thereby fixing the C_{2n} coefficients. As before, the goal is actually more ambitious: each particular graph contributing to $\mathcal{A}_n$ should be $O(p^n)$, so that the power counting is transparent.

As any single diagram in the effective theory is proportional to positive powers of p, computing the leading term $\mathcal{A}_{-1}$ must involve summing an infinite set of diagrams. It is easy to see that the leading term $\mathcal{A}_{-1}$ can be reproduced by the sum of bubble diagrams with C_0 vertices which yields in the MS scheme

$$\mathcal{A}_{-1} = \frac{-C_0}{\left[1 + \frac{C_0 M}{4\pi} ip\right]} \ . \tag{140}$$

Comparing this with eq. (139) gives $C_0 = 4\pi a/M$, as in the previous section. However, there is no expansion parameter that justifies this summation: each individual graph in the bubble sum goes as $C_0(C_0Mp)^L \sim (4\pi a/M)(iap)^L$, where L is the number of loops. Therefore each graph in the bubble sum is bigger than the preceding one, for $|ap| > 1$, while they sum up to something small.

This is an unpleasant situation for an effective field theory; it is important to have an expansion parameter so that one can identify the order of any particular graph, and sum the graphs consistently. Without such an expansion parameter, one cannot determine the size of omitted contributions, and one can end up retaining certain graphs while dropping operators needed to renormalize those graphs. This results in a model-dependent description of the short distance physics, as opposed to a proper effective field theory calculation.

Since the sizes of the contact interactions depend on the renormalization scheme one uses, the task becomes one of identifying the appropriate subtraction scheme that makes the power counting simple and manifest. The MS scheme fails on this point; however this is not a problem with dimensional regularization, but rather a problem with the minimal subtraction scheme itself. A momentum space subtraction at threshold behaves similarly.

Next, consider an alternative regularization and renormalization scheme, namely to using a momentum cutoff equal to Λ. Then for large a one finds $C_0 \sim (4\pi/M\Lambda)$, and each additional loop contributes a factor of $C_0(\Lambda + ip)M/4\pi \sim (1 + ip/\Lambda)$. The problem with this scheme is that for $\Lambda \gg p$ the term ip/Λ from the loop is small relative to the 1, and ought to be ignorable; however, neglecting it would fail to reproduce the desired result eq. (139). This scheme suffers from significant cancellations between terms, and so once again the power counting is not manifest.

Evidently, since $\mathcal{A}_{-1}$ scales as $1/p$, the desired expansion would have each individual graph contributing to $\mathcal{A}_{-1}$ scale as $1/p$. As the tree level contribution is C_0, I must therefore have C_0 be of size $\propto 1/p$, and each additional loop must be $O(1)$. This can be achieved by using dimensional regularization and the PDS (power divergence subtraction) scheme.The PDS scheme involves subtracting from the dimensionally regulated loop integrals not only the $1/(D-4)$ poles corresponding to log divergences, as in MS, but also poles in lower dimension which correspond to power law divergences at $D = 4$. The integral I_n in eq. (130) has a pole in $D = 3$ dimensions which can be removed by adding to I_n the counterterm

$$\delta I_n = -\frac{M(ME)^n \mu}{4\pi(D-3)}, \tag{141}$$

so that the subtracted integral in $D = 4$ dimensions is

$$I_n^{PDS} = I_n + \delta I_n = -(ME)^n \left(\frac{M}{4\pi}\right)(\mu + ip). \tag{142}$$

In this subtraction scheme

$$\mathcal{A} \;=\; -\frac{M}{4\pi}\left[\frac{4\pi}{M\sum C_{2n}p^{2n}}+\mu+ip\right]^{-1} \;. \tag{143}$$

By performing a Taylor expansion of the denominator of the above expression, and comparing with eq. (137), one finds that for $\mu \gg 1/|a|$, the couplings $C_{2n}(\mu)$ scale as

$$C_{2n}(\mu) \sim \frac{4\pi}{M\Lambda^n\mu^{n+1}} \;. \tag{144}$$

Eq. 144 implies that $\mu \sim p$, $C_{2n}(\mu) \sim 1/p^{n+1}$. A factor of ∇^{2n} at a vertex scales as p^{2n}, while each loop contributes a factor of p. The power counting rules for the case of large scattering length are therefore:

1. Each propagator counts as $1/p^2$;
2. Each loop integration $\int \mathrm{d}^4q$ counts as p^5;
3. Each vertex $C_{2n}\nabla^{2n}$ contributes p^{n-1}.

We see that this scheme avoids the problems encountered with the choices of the MS ($\mu=0$) or momentum cutoff ($\mu \sim \Lambda$) schemes. First of all, a tree level diagram with a C_0 vertex is $O(p^{-1})$, while each loop with a C_0 vertex contributes $C_0(\mu)M(\mu+ip)/4\pi \sim 1$. Therefore each term in the bubble sum contributing to $\mathcal{A}_{-1}$ is of order p^{-1}, unlike the case for $\mu = 0$. Secondly, since $\mu \sim p$, it makes sense keeping both the μ and the ip in eq. (142) as they are of similar size, unlike what we found in the $\mu = \Lambda$ case. The PDS scheme retains the nice feature of MS that powers of q inside the loop.

Starting from the above counting rules (proposed in [13,14] and referred to in the literature as "KSW" counting) one finds that the leading order contribution to the scattering amplitude $\mathcal{A}_{-1}$ scales as p^{-1} and consists of the sum of bubble diagrams with C_0 vertices; contributions to the amplitude scaling as higher powers of p come from perturbative insertions of derivative interactions, dressed to all orders by C_0. The first three terms in the expansion are

$$\begin{aligned}
\mathcal{A}_{-1} &= \frac{-C_0}{\left[1+\frac{C_0M}{4\pi}(\mu+ip)\right]}\;,\\
\mathcal{A}_{0} &= \frac{-C_2p^2}{\left[1+\frac{C_0M}{4\pi}(\mu+ip)\right]^2}\;,\\
\mathcal{A}_{1} &= \left(\frac{(C_2p^2)^2M(\mu+ip)/4\pi}{\left[1+\frac{C_0M}{4\pi}(\mu+ip)\right]^3}-\frac{C_4p^4}{\left[1+\frac{C_0M}{4\pi}(\mu+ip)\right]^2}\right)\;,
\end{aligned} \tag{145}$$

where the first two correspond to the Feynman diagrams in Fig. 6. The third term, $\mathcal{A}_1$, comes from graphs with either one insertion of $C_4\nabla^4$ or two insertions of $C_2\nabla^2$, dressed to all orders by the C_0 interaction.

Comparing eq. (145) with the expansion of the amplitude eq. (137), the couplings C_{2n}

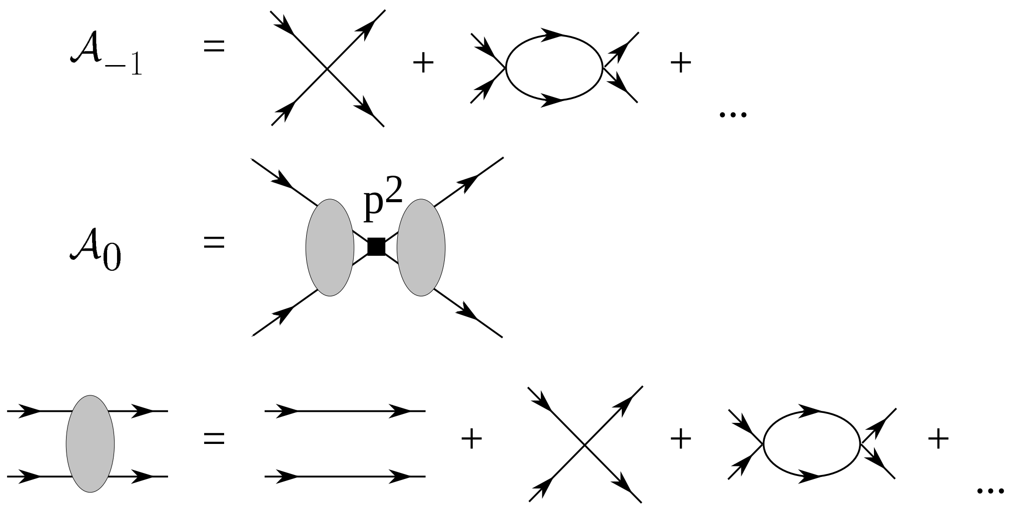

Figure 6: *Leading and subleading contributions arising from local operators. The unmarked vertex is the C_0 interaction, which is summed to all orders; the one marked "p^2" is the C_2 interaction, etc.*

are related to the low energy scattering data a, r_n:

$$\begin{aligned}
C_0(\mu) &= \frac{4\pi}{M}\left(\frac{1}{-\mu+1/a}\right) , \\
C_2(\mu) &= \frac{4\pi}{M}\left(\frac{1}{-\mu+1/a}\right)^2 \frac{r_0}{2} , \\
C_4(\mu) &= \frac{4\pi}{M}\left(\frac{1}{-\mu+1/a}\right)^3 \left[\frac{1}{4}r_0^2+\frac{1}{2}\frac{r_1}{\Lambda^2}\left(-\mu+1/a\right)\right] . \qquad (146)
\end{aligned}$$

Note that assuming $r_n \sim 1/\Lambda$, these expressions are consistent with the scaling law in eq. (144).

4.3.3 Beyond the effective range expansion

So far, we have developed an elaborate machinery to just reproduce the effective range expansion! The payoff comes when one includes electromagnetic and weak interactions. The example I will briefly describe the application of the pionless effective theory to here is the application of the pionless effective theory to radiative capture process $np \to d\gamma$. At leading order, the ingredients to the calculation are the following:

i. One starts with the the nucleon kinetic two-nucleon C_0 interaction for the ${}^3S_1$ channel, written as

$$\mathcal{L} = \ldots - C_0(N^T P_i N)^\dagger (N^T P_i N) \ , \qquad (147)$$

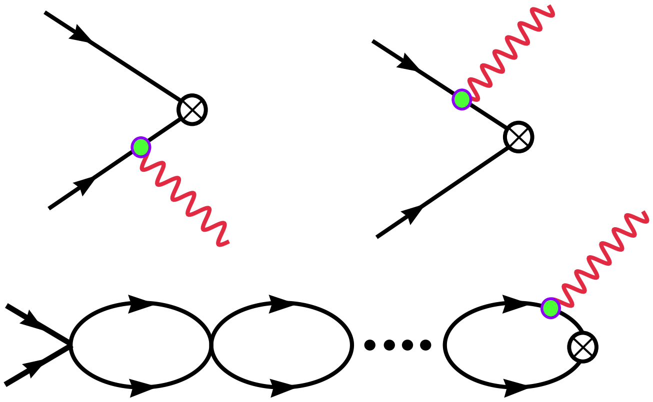

Figure 7: *The leading order contribution to $np \to d\gamma$. Solid lines denote nucleons, wavy lines denote photons. The photon coupling is through the nucleon anomalous magnetic moment operator in $\mathcal{L}_B$; the resummed unmarked vertex is the C_0 interaction. The crossed circle represents and insertion of the deuteron interpolating field $\mathcal{D}_i$. The bubble chain without photon insertions (not shown) is used to compute the wave function renormalization Z, and to fit C_0 to get the correction deuteron binding energy. See [15–17].*

where N is the nucleon doublet, and P_i is the projection operator onto the ${}^3S_1$ channel:

$$P_i = \frac{1}{\sqrt{8}}\sigma_2\sigma_i\tau_2 \ , \quad \mathrm{Tr} P_i P_j = \frac{1}{2}\delta_{ij} \ , \tag{148}$$

where the σ_i act on spin and the τ_i act on isospin.

ii. One uses the convenient interpolating field $\mathcal{D}_i(x) \equiv N^T P_i N(x)$ to be the operator that creates a deuteron at the point x. The coupling C_0 can be fixed by ensuring that the pole in $\mathcal{A}_{-1}$ occurs at the deuteron binding energy. The leading order wave function normalization Z is extracted by looking at the residue at the pole. ($\sqrt{Z}$ is just the amplitude for our operator $\mathcal{D}_i$ to create a physical deuteron.)

iii. $np \to d\gamma$ occurs by emitting a magnetic photon, and so one needs to include in the Lagrangian the anomalous magnetic moment interaction of the nucleons:

$$\mathcal{L}_B = \frac{e}{2M_N} N^\dagger(\kappa_0 + \kappa_1\tau_3)\boldsymbol{\sigma}\cdot\mathbf{B}N \ , \tag{149}$$

where $\kappa_0 = \frac{1}{2}(\kappa_p + \kappa_n)$ and $\kappa_1 = \frac{1}{2}(\kappa_p - \kappa_n)$ are the isoscalar and isovector nucleon magnetic moments with $\kappa_p = 2.79$, $\kappa_n = -1.91$.

iv. Then at leading order one sums up the bubble chain with one insertion of the magnetic moment operator, as shown in Fig. 7.

From these graphs one finds the capture cross section

$$\sigma = \frac{8\pi\alpha\gamma^5\kappa_1^2 a_0^2}{vM_N^5}\left(1 - \frac{1}{\gamma a_0}\right)^2 , \tag{150}$$

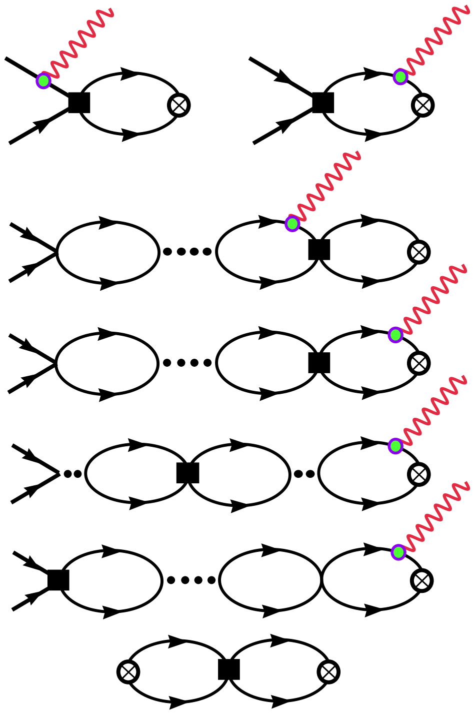

Figure 8: *The graphs contributing to $np \to d\gamma$ at NLO. The black square corresponds to an insertion of a C_2 interaction, the circle to the nucleon anomalous magnetic moment, and the resummed unmarked vertex to the C_0 interaction. The last graph is the contribution to wave function renormalization at this order. Figure from ref. [17].*

where α is the fine structure constant and v is the magnitude of the neutron velocity (in the proton rest frame), $a_0 = -23.714 \pm .013$ fm is the ${}^1S_0$ scattering length and $\gamma \equiv \sqrt{M_N B}$, where b is the deuteron binding energy.. This agrees with old results of Bethe and Longmire when terms in their expression involving the effective range (which are higher order in our expansion) are neglected.

At next-to-leading order ("NLO") one needs to sum all relevant diagrams involving a single insertion of a C_2 vertex (that is, a 2-derivative contact interaction, whose value is fit to the experimental effective range in NN scattering) for both the ${}^1S_0$ and ${}^3S_1$ channels, as in Fig. 8.

However this is not all. At the same order one finds a new contact interaction which cannot be fit to NN scattering data. It is a 2-body interaction with a magnetic photon attached, involving a new coupling constant L_1:

$$\mathcal{L}_{L_1} = eL_1(N^T P_i N)^\dagger (N^T P_3 N) B_i \ . \tag{151}$$

A gauge field is power counted the same as a derivative, and so the B field counts as two spatial derivatives. Thus graphs with one L_1 insertion and in infinite number of C_0 insertions comes in at the same order as $\kappa\ \gamma NN$ vertex summed with an infinite number of C_0 vertices and one C_2 insertion. So at NLO one needs also to include the graph in Fig. 9.

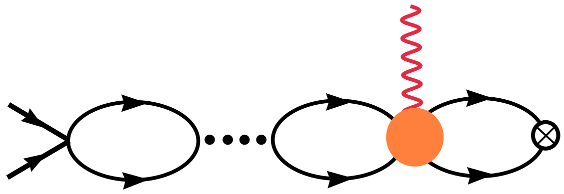

Figure 9: *An additional graph at NLO including an insertion of the L_1 operator. From ref. [17], courtesy of M. Savage.*

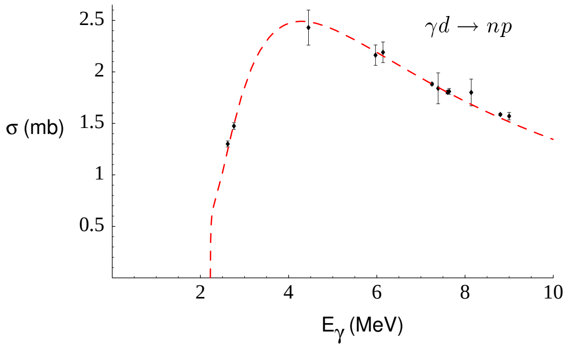

Figure 10: *Cross section for $\gamma d \to np$ breakup as a function of photon energy E_γ. The dashed line is the theoretical calculation of ref. [18], as is this figure, courtesy of G. Rupak. The data are from ref. [19]*

The NLO result deviates from the old effective range calculations, since the L_1 operator is a completely new ingredient, and it also changes the dependency of the answer upon the effective range. This new coupling L_1 can be fit to data at one particular neutron velocity, and then one has a highly accurate prediction for neutron capture at any low velocity. The state of the art is presently an N^4LO calculation for the related breakup process $\gamma d \to np$ by Gautam Rupak. His results are shown in Fig. 10

4.3.4 Few nucleon systems

Extension of the pionless effective theory to systems with more than two nucleons is a very elegant and interesting subject, pioneered by Bedaque, Hammer and Van Kolck [20–22] (for a more recent review, see [23]). A fascinating result of the analysis is that Nd scattering in the $j = \frac{3}{2}$ channel is well described at leading order by summing up two-body interactions, along the lines described in the previous section. However in the $j = \frac{1}{2}$ channel, already at leading order a 3-body contact interaction is needed to renormalize the scattering amplitude. Furthermore, the strength of this interaction exhibited limit-cycle behavior as a function of the momentum cutoff.

Unfortunately, I do not have time to discuss it, but I did want to show one plot from

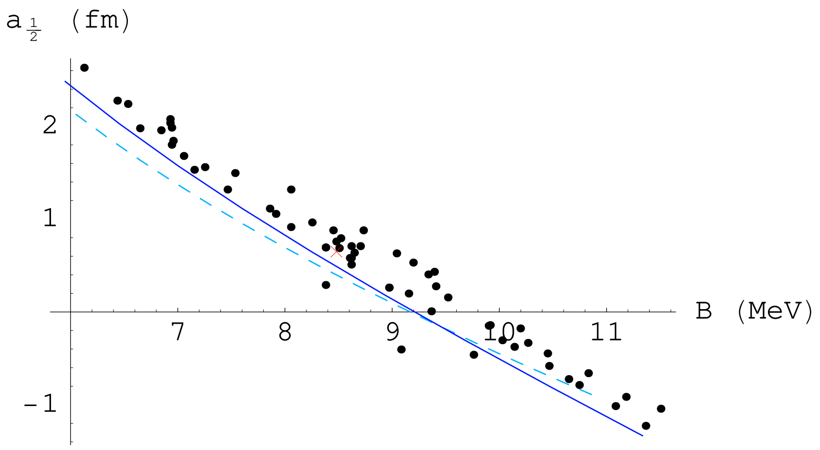

Figure 11: *Correlation between the $j = \frac{1}{2}$ s-wave Nd scattering length and the triton binding energy (Phillips line): predictions of various potential models (black dots), EFT in LO (light dashed line) and NLO (dark solid line), varying the 3-body contact interaction. The experimental value is marked by the red cross. From [23], courtesy of the authors.*

the review, showing the so-called Phillips line, in Fig. 11. Plotted here is a plot of the $j = \frac{1}{2}$ nd scattering length, versus the triton binding energy in MeV. Plotted as black dots are the results from numerous potential models. They evidently fall near a curve, called the "Phillips line". Also plotted are the LO and NLO results from the pionless effective field theory; these calculations require a counterterm for a 3-body operator in order to render the calculations finite. The residual finite part of this interaction must therefore be fit to data. In Fig. 11 the EFT results are shown for a continuous range of this coupling constant for the 3-body force, generating curves which lie close to the black dots. The interpretation is evident: without realizing it, the different potential models have assigned different, and essentially random values to the 3-body force[6], hence the one-parameter spread in results. The figure also makes it clear that by appropriately choosing the value for this 3-body force, the NLO EFT calculation will agree very well with experiment, lying at the closest approach of the solid curve to the red cross. This plot is an excellent advertisement for why effective field theory is a good tool for low energy nuclear physics.

4.4 Including pions in the EFT for nuclear physics

The original suggestion for applying effective field theory to nuclear physics was due to Weinberg [24,25]. His idea was use the chiral Lagrangian for meson-nucleon interactions discussed in the previous lecture, supplemented with multi-nucleon operators. Realizing that the system was nonperturbative, he advocated performing a straightforward chiral expan-

[6]Note that sequential 2-body interactions at short distance can be equivalent to a 3-body interaction when viewed with low resolution. So even if every model had included no fundamental 3-body interaction, since they all have different 2-body interactions at short distance, they would still be scattered over the Phillips line.

sion of the nucleon-nucleon potential to the desired order, and then solving the Shrödinger equation using that potential. In Feynman diagrams, latter step is equivalent to summing up ladder diagrams where the rungs of the ladder are interactions between the nucleons via the potential. For the expansion of the potential, the chiral power counting Weinberg proposed was the usual one, basically just counting inverse powers of Λ or f_π, where $C_0 \sim 1/f_\pi^2$ entered at the same order as one-pion exchange, which has the form

$$\frac{g_A^2}{f_\pi^2}\frac{(\mathbf{q}\cdot\boldsymbol{\sigma})^2}{\mathbf{q}^2+m_\pi^2}\ . \tag{152}$$

It was soon discovered that while the chiral expansion of the potential was well defined, the ladder sum was not as it required counterterms not encountered in the expansion of the potential, and which were arbitrarily high order in the chiral expansion [26]. This is equivalent to saying that Weinberg's method sums up a subset of diagrams arbitrarily high in the chiral expansion. This may be OK if that subset does not affect significantly the final answer, but in that case, why sum the higher order effects at all? If they are important, then one has to justify the exclusion of other terms higher order in the chiral expansion, or else what one has is just another model for nuclear physics, and not a sensible EFT.

To avoid this problem, the KSW power counting scheme was proposed [13, 14], which was introduced in the previous section for the pionless theory; one-pion exchange enters at $O(p^0)$ (as makes sense from the form of eq. (152), which has a q^2 both in the numerator and the denominator) at the same order as the C_2 interaction. Because this is a consistent scheme, results are independent of renormalization scale at any order of the expansion. It is also a theoretically appealing scheme because pions are treated perturbatively, and so calculations may be performed analytically. Unfortunately, the scheme was shown to fail to converge well at fairly low momentum in the $^3S_1$ channel.

Apparently what is happening is that because of the nonperturbative interactions, operators acquire large anomalous dimensions which can cause them to become either much more relevant, or much less relevant than one naively expects. Thus the perturbative analysis of counterterms in ref. [13, 14] can be misleading: the need for a counterterm at high order in the chiral expansion could just signify that the operator in question has a large negative anomalous dimension and is actually *very unimportant* to the calculation.

Currently, one needs to perform numerical calculations to decide. On the one hand, while perturbative ladder diagrams suggest an infinite number of counterterms are needed to renormalize two nucleons interacting via the tensor force in the $^3S_1-{}^3D_1$ channels, numerical results show that only a single counterterm is required. That is good for application of Weinberg's power counting scheme. On the other hand, one finds that in all arbitrarily high partial wave with an attractive tensor interaction, a counterterm is required at leading order, which is not in agreement with Weinberg's expansion scheme.

In conclusion, I would say that EFT for nuclear physics at momentum transfers comparable to the pion mass and higher seems to work pretty well if one follows a patchwork of power counting rules that are derived from numerical experiments. I do not find this totally satisfactory though; some new theoretical ideas for how to better understand the power counting for this nonperturbative EFT would be welcome.

4.5 Trapped atoms

One might think that having particles with an unnaturally large two-body scattering length would be peculiar to nuclear physics. However, atomic physicists trapping collections of atoms can tune the scattering length of atom-atom scattering to be very large by adjusting an external magnetic field. In the limit that the scattering length diverges and the inter-particle spacing is much less than the range of the interactions, there is no dimensionful scale for low energy scattering other than the incoming energy. Therefore any such system should exhibit universal properties, up to trivial rescaling to account for the particle masses. For example, the rescaled specific heat, or critical temperature for pairing should be almost the same for tuned atoms as for a neutron gas.

This is an ideal system for the application of the type of EFT expansion developed for nucleon-nucleon interactions. Unfortunately, it is a nonperturbative many-body problem (an infinite number of particles lie within a scattering length of each other) and not amenable to analytic calculation with any reliability. However, knowing that a two particle contact interaction is all that is needed to describe the system, it is relatively simple to construct a lattice version of the problem [27] which can be simulated numerically [28]. It is the beauty of effective field theory that allows one to extract information about a complicated many-body atomic system by means of a two particle contact interaction, formulated in a spacetime consisting of a discrete set of points!

Problems:

IV.1) Reproduce the result eq. (140).

IV.2) Using the lowest order interaction of eq. (147) and the interpolating field $\mathcal{D}_i = N^T P_i N$, where P_i is the projection operator in eq. (148), relate C_0 to the binding energy B of the deuteron, and find the wave function renormalization Z. Ingredients:

$$G(\overline{E})\,\delta_{ij} = \int \mathrm{d}^4x e^{-i(Et-\mathbf{p}\cdot\mathbf{x})} \langle 0|\mathrm{T}\left[\mathcal{D}_i^\dagger(x)\mathcal{D}_j(0)\right]|0\rangle = \delta_{ij}\frac{i\mathcal{Z}(\overline{E})}{\overline{E}+B+i\varepsilon}\ , \tag{153}$$

is the sum of bubble-chain graphs with an insertion of $\mathcal{D}_i^\dagger$ at one end, and $\mathcal{D}_j$ at the other. By Lorentz invariance, the propagator only depends on the energy in the center of mass frame, namely

$$\overline{E} \equiv E - \frac{\mathbf{p}^2}{4M} + \dots\ , \qquad E \equiv (p^0 - 2M), \tag{154}$$

where the ellipses refers to relativistic corrections to the dispersion relation. The numerator $\mathcal{Z}$ in eq. (153) is assumed to be smooth near the deuteron pole, and when evaluated at the pole gives the wave function renormalization Z,

$$\mathcal{Z}(-B) \equiv Z = -i\left[\frac{\mathrm{d}G^{-1}(\overline{E})}{\mathrm{d}E}\right]^{-1}_{\overline{E}=-B}. \tag{155}$$

The results are found in the appendix of ref. [15].

5 The effective theory for color superconductivity

5.1 Landau liquid versus BCS instability

A condensed matter system can be a very complicated environment; there may be various types of ions arranged in some crystalline array, where each ion has a complicated electron shell structure and interactions with neighboring ions that allow electrons to wander around the lattice. Nevertheless, the low energy excitation spectrum for many diverse systems can be described pretty well as a "Landau liquid", whose excitations are fermions with a possibly complicated dispersion relation but no interactions. Why this is the case can be simply understood in terms of effective field theories, modifying the scaling arguments to account for the existence of the Fermi surface.

Let us assume that the low energy spectrum of the condensed matter system has fermionic excitations with arbitrary interactions above a Fermi surface characterized by the fermi energy ϵ_F ; call them "quasi-particles". Ignoring interactions at first, the action can be written as

$$S_{free} = \int dt \int d^3p \sum_{s=\pm\frac{1}{2}} \left[\psi_s(p)^\dagger i\partial_t \psi_s(p) - (\epsilon(p) - \epsilon_F)\psi_s^\dagger(p)\psi_s(p)\right] \tag{156}$$

where an arbitrary dispersion relation $\epsilon(p)$ has been assumed.

To understand how important interactions are, we wish to repeat some momentum space version of the scaling arguments I introduced in the first lecture. In the present case, a low energy excitation corresponds to one for which $(\epsilon(p) - \epsilon_F)$ is small, which means that $\mathbf{p}$ must lie near the Fermi surface. So in momentum space, we will want our scaling variable to vary the distance we sit from the Fermi surface, and not to rescale the overall momentum $\mathbf{p}$. After all, here a particle with $\mathbf{p} = 0$ is a high energy excitation.

This situation is a bit reminiscent of HQET where we wrote $p_\mu = mv_\mu + k_\mu$, with k_μ being variable that is scaled, measuring the "off-shellness" of the heavy quark. So in the present case we will write the momentum as

$$\mathbf{p} = \mathbf{k} + \boldsymbol{\ell} \tag{157}$$

where $\mathbf{k}$ lies on the Fermi surface and $\boldsymbol{\ell}$ is perpendicular to the Fermi surface (shown in Fig. 12 for a spherical Fermi surface). Then $\boldsymbol{\ell}$ is the quantity we vary in experiments and so we define the dimension of operators by how they must scale so that the theory is unchanged when we change $\boldsymbol{\ell} \to r\boldsymbol{\ell}$. If an object scales as r^n, then we say it has dimension n. Then $[k] = 0$, $[\ell] = 1$, and $[\int d^3p = \int d^2k d\ell] = 1$. And if we define the Fermi velocity as $\mathbf{v}_F(\mathbf{k}) = \boldsymbol{\nabla}_k \epsilon(\mathbf{k})$, then for $\ell \ll k$,

$$\epsilon(\mathbf{p}) - \epsilon_F = \boldsymbol{\ell} \cdot \mathbf{v}_F(\mathbf{k}) + \mathcal{O}(\ell^2) \ , \tag{158}$$

and so $[\epsilon - \epsilon_f] = 1$ and $[\partial_t] = 1$. Given that the action eq. (156) isn't supposed to change under this scaling,

$$[\psi] = -\frac{1}{2} \ . \tag{159}$$

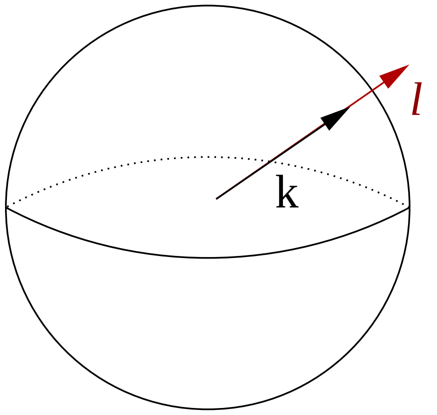

Figure 12: *The momentum* $\mathbf{p}$ *of an excitation is decomposed as* $\mathbf{p} = \mathbf{k} + \boldsymbol{\ell}$*, where* $\mathbf{k}$ *lies on the Fermi surface, and* $\boldsymbol{\ell}$ *is perpendicular to the Fermi surface. Small* $|\boldsymbol{\ell}|$ *corresponds to a small excitation energy.*

Now consider an interaction of the form

$$S_{int} = \int dt \int \prod_{i=1}^{4} (d^2\mathbf{k}_i d\ell_i)\delta^3(\mathbf{P}_{tot})C(\mathbf{k}_1, \ldots, \mathbf{k}_4)\psi_s^\dagger(\mathbf{p}_1)\psi_s(\mathbf{p}_2)\psi_{s'}^\dagger(\mathbf{p}_3)\psi_{s'}(\mathbf{p}_4) \ . \quad (160)$$

This will be relevant, marginal or irrelevant depending on the dimension of C. Apparently $[\delta^3(\mathbf{P}_{tot})C] = -1$. So how does the δ function scale? For generic $\mathbf{k}$ vectors, $\delta(\mathbf{P}_{tot})$ is a constraint on the $\mathbf{k}$ vectors that doesn't change much as one changes ℓ, so that $[\delta^3(\mathbf{P}_{tot})] = 0$. It follows that $[C] = -1$ and that the four fermion interaction is irrelevant...and that the system is adequately described in terms of free fermions (with an arbitrary dispersion relation). This is why Landau liquid theory works and is related to why in nuclear physics Pauli blocking allows a strongly interacting system of nucleons to have single particle excitations.

This is not the whole story though, or else superconductivity would never occur. Let us look more closely at the assumption above $[\delta^3(\mathbf{P}_{tot})] = 0$. Consider the case when all the $\boldsymbol{\ell}_i = 0$, and therefore the $\mathbf{p}_i = \mathbf{k}_i$ and lie on the Fermi surface. Suppose we fix the two incoming momenta $\mathbf{k}_1$ and $\mathbf{k}_2$. The $\delta^3(\mathbf{P}_{tot})$ then constrains the sum $\mathbf{k}_3 + \mathbf{k}_4$ to equal $\mathbf{k}_1 + \mathbf{k}_2$, which generically means that the vectors $\mathbf{k}_3$ and $\mathbf{k}_4$ are constrained up to point to opposite points on a circle that lies on the Fermi surface (Fig. 13b). Thus one free parameter remains out of the four independent parameters needed to describe the vectors $\mathbf{p}_3$ and $\mathbf{p}_4$. So we see that in this generic case, $\delta^3(\mathbf{P}_{tot})$ offers three constraints, even when $\boldsymbol{\ell}_i = 0$. Therefore $\delta^3(\mathbf{P}_{tot}) = \delta^3(\mathbf{K}_{tot})$ is unaffected when $\boldsymbol{\ell}$ is scaled, and we find the above assumption $[\delta^3(\mathbf{P}_{tot})] = 0$ to be true, and Landau liquid theory is justified.

However now look at the special case when the collisions of the incoming particles are nearly head-on, $\mathbf{k}_1 + \mathbf{k}_2 = 0$. Now $\delta^3(\mathbf{P}_{tot})$ constrains the outgoing momenta to satisfy $\mathbf{k}_3 + \mathbf{k}_4 = 0$. But as seen in Fig. 13a, this only constrains $\mathbf{k}_3$ and $\mathbf{k}_4$ to lie on opposite sides of the Fermi surface. Thus $\delta^3(\mathbf{P}_{tot})$ seems to be only constraining two degrees of freedom, and could be written as $\delta^2(\mathbf{k}_3+\mathbf{k}_4)\delta(0)$. This singularity obviously arose because the set the

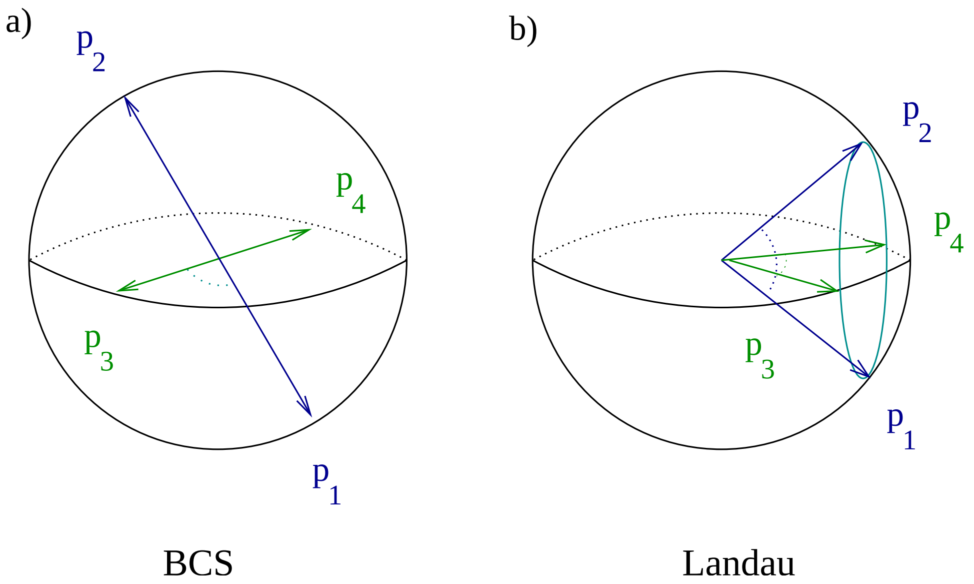

Figure 13: *Fermions scattering near the Fermi surface. (a) Head-on collisions: With* $\mathbf{k}_1+\mathbf{k}_2 = 0$*, only two degrees of freedom in the outgoing momenta* $\mathbf{k}_3$ *and* $\mathbf{k}_4$ *are constrained, as they can point to any two opposite points on the Fermi surface. (b) The generic Landau liquid case, where the incoming particles do not collide head-on, and three degrees of freedom in the outgoing momenta* $\mathbf{k}_3$ *and* $\mathbf{k}_4$ *are constrained, as they must point to opposite sides of a particular circle on the Fermi surface. Figure from ref. [29], courtesy of Thomas Schäfer.*

$\boldsymbol{\ell}_i = 0$. For nonzero $\boldsymbol{\ell}$ the $\delta(0)$ becomes $\delta(\boldsymbol{\ell}_{tot})$, and as a result, the δ function does scale with $\boldsymbol{\ell}$: $[\delta^3(\mathbf{P}_{tot}(] = -1$. But then $[C] = 0$ for head-on collisions, and the interaction is marginal! Quantum corrections either make it either irrelevant or relevant; it turns out that for an attractive interaction, the interaction becomes relevant, and for a repulsive interaction, it becomes irrelevant, just as we found for a δ-function interaction in two dimensions.

Therefore, an attractive contact interaction between quasiparticles becomes strong exponentially close to the Fermi surface (since the coupling runs logarithmically), and can lead to pairing and superconductivity just as the asymptotically free QCD coupling leads to quark condensation and chiral symmetry breaking. The BCS variational calculation shows that the pairing instability does indeed occur; the effective field theory analysis explains why Cooper pairs are exponentially large compared to the lattice spacing in superconductors.

5.2 Dense quark matter

With a tree-level one-gluon exchange, two quarks transforming as a color triplet will feel an attractive interaction in the color $\bar{3}$ channel, and a repulsive interaction in the 6 channel. Since superconductivity is generic in any dense fermion system with an attractive interaction, it is clear that such a phase should occur for dense quark matter. Just as electromagnetism is spontaneously broken by the condensation of electron pairs in an ordinary superconductor, color will be broken when quarks condense. Color superconductivity was first discussed in the early 1980's [30]. In the past decade there has been a resurgence of interest in dense quark matter and color superconductivity, sparked by the papers [31–34]. Two interesting features that were discovered was that the gap is parametrically larger in a color superconductor than in an ordinary metal [34], and that the ground state of dense QCD with three massless flavors spontaneously breaks chiral symmetry, even though at high density the $\langle \bar{q}q \rangle$ condensate is expected to vanish. As a result, there are nine Goldstone bosons in the theory (eight for flavor, one for baryon number). Thus there is no gap in the spectrum of the theory in many quantum number channels, and so the phase structure becomes very rich when quark masses are turned on and charge neutrality is enforced. An effective field theory is indispensable for understanding these light modes and the phase structure.

I will be assuming a very high quark number chemical potential, $\mu \gg \Lambda_{QCD}$ so that perturbative QCD applies. I will simultaneously be assuming that $m_c > \mu$ so that I can restrict the discussion to three flavors. It is certain that there is no place in the universe where these conditions are met! At best, one can hope that there is a quark matter star somewhere, and that even though its chemical potential will be $O(\Lambda_{QCD})$, qualitative features discovered at large chemical potential will still hold in the nonperturbative regime[7]. So the subject may be purely academic, but we're academics, after all, and it is fun.

Just as electron pairs condense in ordinary superconductivity, quark pairs are expected to condense in the attractive color $\bar{3}$ channel. However, if the u, d and s quarks are massless, their Fermi surfaces match up, and quarks of different flavors can condense with each other. A Lorentz singlet condensate of two left-handed quarks in the most attractive color channel will transform as a $(\bar{3}, 1, \bar{3})$ under $SU(3)_L \times SU(3)_R \times SU(3)_c$, while a condensate of right-handed quarks will transform as a $(1, \bar{3}, \bar{3})$. These are 3×3 matrices, so where are the nonzero entries? The favored phase is to have these matrices just be proportional to the unit matrix; it is called the "Color Flavor Locked" phase, or CFL for short.

Thus the "order parameter" for this phase may be represented as

$$\epsilon_{abc}\epsilon^{ijk}\langle q^a_{L,i} C q^b_{L,j}\rangle = -\epsilon_{abc}\epsilon^{ijk}\langle q^a_{R,i} C q^b_{R,j}\rangle = \Lambda^3 \delta^k_c \ , \tag{161}$$

where a, b, c are $SU(3)_c$ indices, and i, j, k are flavor indices. Note that flavor and color have become correlated as the above condensates vanish if $c \neq k$. (I prefer a Weyl fermion basis for keeping track of the quantum numbers; see problem (V.1)). This condensate breaks $SU(3)_L \times SU(3)_R \times SU(3)_c$ down to the diagonal $SU(3)$.

Actually the condensate above is written in the usual sloppy way we talk about the Higgs mechanism. In fact, gauge variant operators always have zero expectation value unless one gauge fixes. The actual gauge invariant order parameters are four (and more) quark operators, such as

$$(\epsilon^{ijk}\epsilon_{mnp})(\epsilon_{abc}\epsilon^{dec})\langle q^a_{R,i} C q^b_{R,j} \left(q^d_{L,m} C q^e_{L,n}\right)^\dagger\rangle = \Lambda^6 \delta^k_p \ . \tag{162}$$

[7]Even should this be true, it seems not too likely that we will ever get enough data on such a star to really test the theory. My hope is that some day people will figure out how to simulate lattice QCD at finite chemical potential and we can do experiments on a computer. If you want to make a truly major contribution to nuclear physics, solving that problem would be a fine choice!

This color neutral order parameter transforms as a $(3, \bar{3})$ under $SU(3)_L \times SU(3)_R$, and acquires a diagonal vev, breaking $SU(3)_L \times SU(3)_R$ down to $SU(3)_V$...just like the $\langle \bar{q}q \rangle$ condensate of eq. (53) at zero chemical potential. The fermionic excitations in this ground state are "gapped": it requires a minimum energy 2Δ to produce a particle-hole pair, where Δ is called the gap, and $\Lambda^3 \propto \Delta$ in the above equation.

Although sloppy, the advantage of eq. (161) is that it indicates that $SU(3)_c$ has been higgsed, and that the gluons have become massive. Why is this significant, since at finite density, one already has Debye screening, which means that color electric fields fall off from sources as if the gluon had a mass $m = g\mu$? The difference is that without the Higgs effect, magnetic fields at nonzero frequency would have power law fall-off, and not be screened. In the color superconductor, though, magnetic fields are screened as well. This means that it is self consistent to treat calculate properties of this state in perturbation theory, since the relevant is the running coupling α_s evaluated at the scale of the chemical potential μ, which is small for sufficiently high μ.

The spectrum of the theory now has gluons with mass $g\mu$; fermionic excitations around the Fermi surface with mass Δ, where $\ln \Delta/\mu = O(1/g)$ [34]; and eight massless Goldstone bosons. This allows one to construct a series of effective field theories. First one integrates out degrees of freedom at the scale μ — for example, quark anti-quark pairs which have at least an energy μ, since the quark has to be created above the Fermi sea. Next one integrates out the gluons at scale $g\mu$; then the quasiparticle excitations at scale Δ. One is left with a chiral Lagrangian for the Goldstone bosons, where the coefficients are calculable.

We start with the QCD Lagrangian,

$$\mathcal{L} = \bar{q}(i\not{D} + \mu\gamma^0)q - \bar{q}_L M q_R - \bar{q}_R M^\dagger q_L - \frac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu} , \tag{163}$$

where I have reintroduced the quark mass matrix M. Since we wish to consider quarks near the Fermi surface, first consider free quarks with a chemical potential μ. Then the Dirac equation reads

$$(\boldsymbol{\alpha} \cdot \mathbf{p} - \mu)\,\psi_\pm = E_\pm \psi_\pm , \qquad (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\psi_\pm = \pm\psi_\pm , \tag{164}$$

where $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$ and $E_\pm = -\mu \pm p$. We see that for $p \sim p_F = \mu$, the ψ_+ with $E_+ \sim 0$ correspond to states near the Fermi surface and ψ_-, with $E_- \sim -2\mu$, to states far from it. We therefore follow Hong [35, 36] and define for the interacting theory

$$\psi_\pm(\mathbf{v_F}, x) = e^{ip_F v_\mu x^\mu} \left(\frac{1 \pm \boldsymbol{\alpha} \cdot \hat{\mathbf{v}}_\mathbf{F}}{2} \right) q , \tag{165}$$

where $v_\mu = (1, \mathbf{v_F})$ and $\mathbf{v_F}$ is the Fermi velocity we are expanding about, with $\mathbf{k} = p_F \hat{\mathbf{v}}_\mathbf{F}$ corresponding to the vector $\mathbf{k}$ in Fig. 12. The prefactor removes the rapid phase common to all fermions in the vicinity of this patch of the Fermi surface specified by Fermi velocity $\mathbf{v_F}$.

One then constructs a $1/p_F$ expansion of this theory, integrating out the ψ_- fields and the hard gluons (whose propagators are $1/q^2 \leq 1/p_F^2$). At tree level, integrating out the π_- fields is equivalent to replacing them by their equations of motion,

$$\psi_{-,L} = \frac{1}{2p_F}\left(i\boldsymbol{\alpha}_\perp \cdot \mathbf{D}\psi_{+,L} + \gamma^0 M \psi_{+,R} \right) . \tag{166}$$

where $\boldsymbol{\gamma}_{\|} \equiv \hat{\mathbf{v}}_F(\hat{\mathbf{v}}_F \cdot \boldsymbol{\gamma})$ and $\boldsymbol{\gamma}_{\perp} = (\boldsymbol{\gamma} - \boldsymbol{\gamma}_{\|})$. Integrating out the hard gluons generates a four quark vertex which is the attraction that gives rise to quark condensation; this in term introduces a "gap" term in the effective theory, reflecting that we are not expanding about the perturbative vacuum, but one with symmetry breaking; the gap Δ is solved for self consistently. I highly recommend the various papers and reviews by Thomas Schäfer on the subject if you wish to understand the procedure in detail [29, 37–40].

The effective theory to $O(1/p_F)$ takes the form (see for example [41, 42])

$$\begin{aligned}\mathcal{L} &= \psi^{\dagger}_{L+}(iv\cdot D)\psi_{L+} - \frac{\Delta}{2}\left(\psi^{ai}_{L+}C\psi^{bj}_{L+}\left(\epsilon_{abc}\epsilon_{ijk}(X^{\dagger})^{ck}\right) + \text{h.c.}\right)\\ &\quad - \frac{1}{2p_F}\psi^{\dagger}_{L+}\left((\not{D}_{\perp})^2 + MM^{\dagger}\right)\psi_{L+} + \left(R\leftrightarrow L, M\leftrightarrow M^{\dagger}, X\leftrightarrow Y\right) + \ldots \quad (167)\end{aligned}$$

where $D_\mu = \partial_\mu + igA_\mu$. The $SU(3)$ matrices X and Y are spacetime dependent, and parametrize the wiggles of the $\langle q_L q_L\rangle$ and $\langle q_R q_R\rangle$ condensates respectively. Under $SU(3)_L\times SU(3)_R\times SU(3)_c$ they transform as $X = (3,1,3)$ and $Y = (1,3,3)$, and they can be written in the form $X = e^{i(\Pi+\pi)/f}$ and $Y = e^{i(\Pi-\pi)/f}$. The Π fields can be rotated away by an $SU(3)_c$ color transformation—this is called "choosing unitary gauge", and the gluons have a mass gf in this gauge. The π fields are the octet that arises from $SU(3)_L\times SU(3)_R$ breaking, and we can define the $SU(3)_c$ neutral matrix $\Sigma = XY^{\dagger}$ which transforms as $(3,\bar{3},1)$ just like the Σ matrix we discussed in lecture 3, which parametrizes the QCD groundstate at $\mu = 0$.

In the above Lagrangian, the $MM^{\dagger}$ term is called the Bedaque-Schäfer term [43] and it is crucial to understanding the fate of the CFL phase as one turns on nonzero quark masses; the discovery of its existence and dramatic consequences is one of the triumphs of the effective field theory approach to color superconductivity. It arises from the mass dependence of ψ_- in eq. (166). What Bedaque and Schäfer realized is that the combination $\mu^{(BS)}_L \equiv -MM^{\dagger}/(2p_f)$ enters the effective theory like the time component of an $SU(3)_L$ gauge field would, if one gauged left-handed flavor symmetries. Similarly, $\mu^{(BS)}_R \equiv -M^{\dagger}M/(2p_F)$ enters like an $SU(3)_R$ gauge field. This fake gauge invariance therefore will constrain how the Bedaque-Schäfer terms enter the low energy chiral Lagrangian.

At $O(1/p_F^2)$, the high density effective theory contains four-fermion interactions which contain two powers of M and others that contain an M and an $M^{\dagger}$; see [41].

5.3 The chiral Lagrangian for the CFL phase

We are now ready to construct the chiral Lagrangian for the octet of Goldstone bosons in the CFL phase. There is an excellent reason to do so: the CFL order parameter eq. (162) is perfectly symmetric in flavor because we assumed massless quarks. In the real world, the strange quark mass m_s could be comparable or bigger than the gap Δ even for rather large chemical potential μ, since Δ is exponentially smaller than μ. That means that the fermi surface of free strange quarks would not be congruent with the fermi surface of the u and d quarks, and pairing between them would be cost energy. Thus as the strange quark mass is turned on, the vacuum comes under stress, and eventually phase transitions are expected. If the phase transitions are second order or weakly first order (with latent heat less than Δ), then they will have to involve light degrees of freedom, and should be visible in the chiral Lagrangian. In fact, there are a number of interesting phase transitions one can find as m_s is turned up, starting with kaon condensation.

The cutoff of the chiral Lagrangian will be Δ, the scale of the quasiparticles we have integrated out (they are like the baryons in the usual chiral Lagrangian discussed in the third lecture). There several basic differences between the CFL chiral Lagrangian, and the usual one:

i. Since the chemical potential violates Lorentz invariance, the Lagrangian will not respect Lorentz invariance either; the speed of light will be replaced by the speed of sound, which to leading order in perturbation theory is $c/\sqrt{3}$, the result for a weakly interacting relativistic gas.

ii. Baryon number is spontaneously broken as well as $SU(3) \times SU(3)$, so there will be a corresponding Goldstone boson B, the "superfluid mode".

iii. At $\mu = 0$ we ignore the $U(1)_A$ symmetry, as it is badly broken by instanton effects. At high density, the instantons do not play a role, so there is a light η', the Goldstone boson for broken $U(1)_A$ symmetry, and the Σ field carries $U(1)_A$. Assign a $U(1)_A$ charge $Q_A = +1$ to q_L and $Q_A = -1$ to q_R. Then the order parameter eq. (162) carries $Q_A = -4$, and so must the Σ field. On the other hand, the QCD Lagrangian is invariant of the quark mass M carried $Q_A = 2$. Therefore Σ cannot couple to odd powers of M. This assumes that the $\mu = 0$ condensate $\langle \bar{q}q \rangle$ vanishes at high density, an assumption on solid ground.

iv. As we derived in the previous section, in the EFT below μ, the Bedaque-Schäfer term μ_{BS} appears as the time component of an $SU(3)_V$ flavor gauge field; therefore it must also in the chiral Lagrangian. Instead of $\partial_0 \Sigma$, we must write $D_0\Sigma = \partial_0 \Sigma + i\mu_L^{(BS)}\Sigma - i\Sigma\mu_R^{(BS)}$.

v. The pion decay constant has been calculated and is $O(\mu)$. Thus $f_\pi \gg \Delta$, unlike in the $\mu = 0$ chiral Lagrangian, where $f_\pi \sim \Lambda/4\pi$. Therefore loop graphs are unimportant to leading order in the small quantity Δ/μ.

Ignoring the B Goldstone boson, as well as the η', and considering only spatially constant Goldstone boson fields, the chiral Lagrangian takes the form

$$\begin{aligned} \mathcal{L} &= f_\pi^2 \left[\frac{1}{4} \mathrm{Tr} D_0 \Sigma D_0 \Sigma^\dagger + \frac{a}{2} \mathrm{Tr} \tilde{M} \left(\Sigma + \Sigma^\dagger \right) + \frac{b}{2} \mathrm{Tr} Q \Sigma Q \Sigma^\dagger \right] \\ & D_0 \Sigma = \partial_0 \Sigma - i \left[\left(\mu_Q Q + \mu_L^{(BS)} \right) \Sigma - \Sigma \left(\mu_Q Q + \mu_R^{(BS)} \right) \right] . \end{aligned} \tag{168}$$

The decay constant f_π has been computed previously [42]. Q is the electric charge matrix $\mathrm{diag}(2/3, -1/3, -1/3)$ while $\mu_{L,R}^{(BS)}$ are the Bedaque-Schäfer terms: $\mu_L^{(BS)} = -\frac{MM^\dagger}{2\mu}$, $\mu_R^{(BS)} = -\frac{M^\dagger M}{2\mu}$. I have included a chemical potential for electric charge,m_Q, since in dense matter, such as the core of stars, there would be the constraint of charge neutrality.

The mass term above has been written in terms of

$$\tilde{M} = M^{-1} \det(M) = \begin{pmatrix} m_d m_s & & \\ & m_u m_s & \\ & & m_u m_d \end{pmatrix} . \tag{169}$$

One can show that this is the only possible form for the coupling of Σ to M at leading order. It is second order in M as required by the $U(1)_A$ symmetry, it transforms as a $(\bar{3},3)$, and it has the property that it vanishes if any two quarks are massless (See problem V.5).

The coefficient a has been computed and is given by $a = 3\frac{\Delta^2}{\pi^2 f_\pi^2}$ [42]. The b term accounts for electromagnetic corrections to the charged meson masses; it has not been calculated but is estimated to be of size $b \sim \frac{\alpha}{4\pi}\Delta^2$.

The meson masses in terms of the parameters a, b are

$$\begin{aligned} m_{\pi^-}^2 &= a(m_u + m_d)m_s + b \\ m_{K^-}^2 &= a(m_u + m_s)m_d + b \\ m_{K^0}^2 &= a(m_d + m_s)m_u \,. \end{aligned} \tag{170}$$

The chiral expansion is in powers of p/Δ, and the Bedaque-Schäfer term appears in the covariant derivative, the theory breaks down when $\mu^{(BS)} \gtrsim \Delta$, or when any one of the quark masses satisfies $m^2 > 2\mu\Delta$. Why is that? Recall that for free fermions near the Fermi surface, the energy is given by $\sqrt{p_F^2 + m^2} = p_F + m^2/2p_F + \ldots$. A chemical potential for baryon number ensures that each quark has the same Fermi *energy* while BCS pairing between states at opposite sides of the Fermi surface requires that the two quarks have the same Fermi *momentum.* Since the pairing gains an energy Δ per quark, while maintaining equal p_F between a heavy quark and a light quark costs $m^2/2p_F$, the pairs will break when $m^2/2p_F \gtrsim \Delta$. Since $\mu = p_F + O(m^2)$, this is the same as saying that the CFL state breaks down completely at $m^2 \gtrsim 2\mu\Delta$. Note that at the breakdown point $m^2 = 2\mu\Delta$, the meson masses are $M^2 \sim \Delta^3/\mu \ll \Delta$, so they are still very light compared to the cutoff of the theory.

5.4 Kaon condensation on top of CFL

But is the CFL ground state stable all the way up to strange quark mass $m_s^2 = 2\mu\Delta$? The answer is no. Just as we saw in a hadronic description of finite density QCD, kaons can alter the ground state to relieve the stress caused by not having the ideal strangeness. However, in that case the problem was that we started from a state with zero strangeness (neutrons) and the system wanted to populate strange quarks, and so we saw evidence for K^- condensation. In the CFL groundstate there are equal numbers of u, d and s, so as m_s is turned on, the system will adjust to reduce its strange quark number, via K^0 condensation.

Unlike the mesons of the chiral Lagrangian we studied in the third lecture, these mesons have chemical potentials. Since we choose a basis where $M = M^\dagger$, the chemical potentials due to μ_Q and $\mu^{(BS)}$, where μ_Q is a real chemical potential added to ensure charge neutrality, while the Bedaque-Schäfer term is a dynamical term reflecting that mesons want to rearrange the CFL ground state if quark masses are unequal. By expanding the kinetic term to quadratric order in the mesons, one can read off their individual chemical potentials. The effective chemical potentials vanish for the π^0, η and η', while for the π^+, K$^+$ and K^0 mesons they are

$$\tilde{\mu}_{\pi^+} = \mu_Q + \frac{m_d^2 - m_u^2}{2\mu} \ , \quad \tilde{\mu}_{K^+} = \mu_Q + \frac{m_s^2 - m_u^2}{2\mu} \ , \quad \tilde{\mu}_{K^0} = \frac{m_s^2 - m_d^2}{2\mu} \ . \tag{171}$$

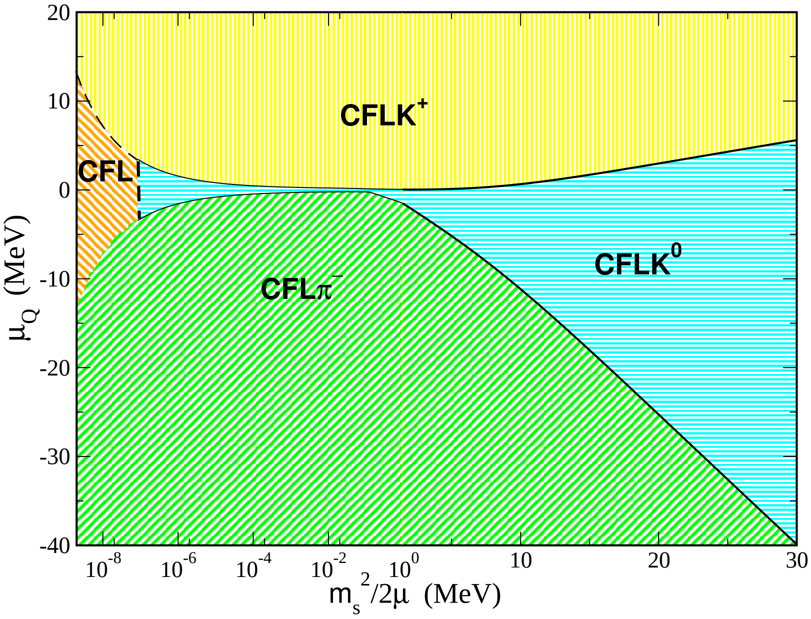

Figure 14: *Meson condensed phases in the neighborhood of the symmetric CFL state are shown in the* $(m_s^2/2\mu) - \mu_Q$ *plane, where* m_s *is the strange quark mass (set to* 150 *MeV),* μ *is the quark number chemical potential, and* μ_Q *is the chemical potential for positive electric charge. At five times nuclear density* $\mu \sim 400$ *MeV and* $(m_s^2/2\mu) \sim 25$ *MeV. Solid and dashed lines indicate first- and second-order transitions respectively. From ref. [44].*

Note that as the strange quark is increased, so is the chemical potential for the K^+ and K^0. The fundamental K^+ and K^0 mesons are made of $\bar{s}u$ and $\bar{s}d$ quarks; these CFL mesons are better thought of as u-quark/s-hole and d-quark/s-hole bound states. So it makes sense that as m_s is increased, it becomes more energetically feasible to turn an s quark into a u or d quark, and the most efficient way to do so is to create a kaon.

Recall that Bose-Einstein condensation occurs when a meson's chemical potential exceeds its mass. Thus for $\mu_Q = 0$, a sufficiently large quark mass will lead to a second order phase transition in the form of kaon condensation [43–45]. Since electromagnetic corrections make the K^+ heavier than the K^0, at $\mu_Q = 0$ one would expect K^0 condensation. However, if $\mu_Q > 0$, positively charged mesons are favored and would get K^+ condensation. Conversely, for sufficiently negative μ_Q the system wants negative charge; K^-'s are too costly as they *add* a strange quark, and so the heavier π^- could condense.

The stationary points of the free energy Ω with respect to variations of the meson fields are found as solutions to the matrix equation

$$\Big[\tilde{\mu}\Sigma^\dagger\tilde{\mu}\Sigma - a\, M\Sigma - b\, Q\Sigma^\dagger Q\Sigma\Big] - h.c. = 0\,. \tag{172}$$

There are distinct solutions for K^0, K^+ and π^- condensation:

$$
\begin{aligned}
\Omega_{\pi^\pm} &= -\tfrac{f_\pi^2}{2}(\tilde{\mu}_{\pi^\pm}^2 - b)(1-\cos\theta_{\pi^\pm})^2 \ , \qquad \cos\theta_{\pi^\pm} = \begin{cases} 1 & M_{\pi^\pm}^2 \geq \tilde{\mu}_{\pi^\pm}^2 \\ \frac{M_{\pi^\pm}^2 - b}{\tilde{\mu}_{\pi^\pm}^2 - b} & M_{\pi^\pm}^2 \leq \tilde{\mu}_{\pi^\pm}^2 \end{cases} \\
\Omega_{K^\pm} &= -\tfrac{f_\pi^2}{2}(\tilde{\mu}_{K^\pm}^2 - b)(1-\cos\theta_{K^\pm})^2 \ , \qquad \cos\theta_{K^\pm} = \begin{cases} 1 & M_{K^\pm}^2 \geq \tilde{\mu}_{K^\pm}^2 \\ \frac{M_{K^\pm}^2 - b}{\tilde{\mu}_{K^\pm}^2 - b} & M_{K^\pm}^2 \leq \tilde{\mu}_{K^\pm}^2 \end{cases} \\
\Omega_{K^0} &= -\tfrac{f_\pi^2}{2}\tilde{\mu}_{K^0}^2(1-\cos\theta_{K^0})^2 \ , \qquad \cos\theta_{K^0} = \begin{cases} 1 & M_{K^0}^2 \geq \tilde{\mu}_{K^0}^2 \\ \frac{M_{K^0}^2}{\tilde{\mu}_{K^0}^2} & M_{K^0}^2 \leq \tilde{\mu}_{K^0}^2 \end{cases}
\end{aligned}
\tag{173}
$$

where in each case Ω measures the free energy relative to the $SU(3)$ symmetric CFL ground state. Evidently the nontrivial solutions with $\theta \neq 0$ represent phases with lower free energy than the symmetric CFL phase. Which one has lower free energy depends on the values for $\mu^{(BS)}$ and μ_Q. The phase diagram one finds is shown in Fig. 14.

One might wonder how similar the kaon condensed phase is to the CFL phase. The number density of strange quarks participating in BCS pairing in the CFL phase is $n_s = O(\mu^2\Delta)$, the volume of a shell of thickness Δ at the Fermi surface. On the other hand, the strange quark density in the K^0 condensate and be computed from the chiral Lagrangian, and one finds $n_s = -\mu_K f_\pi^2 \sin\theta_{K^0}$. As $m_s^2 \to \mu\Delta$, $\sin\theta_{K^0} \to 1$ (maximal condensation) and $n_s = -O(\mu^\Delta)$, which means that as one turns on m_s, by the time one is at the point where CFL is expected to break down, the number of strange quarks participating in pairing has been significantly depleted and the ground state looks quite different than the $SU(3)$ symmetric CFL ground state.

Is kaon condensation the only instability one sees for $m_s^2/\mu\Delta < 1$? Note that $\tilde{M}_{33} \propto m_u m_d$ in eq. (169) is extremely small, and therefore small perturbations could cause $\Sigma_{33} \neq 1$ in the ground state. Indeed, when the η' is included in the chiral Lagrangian, one finds such an instability for condensation of a linear combination of the η and η'. Thus at $\mu_Q = 0$ one finds the additional phase structure shown in Fig. 15.

5.5 Still more phase structure for large m_s

What happens for higher values of $m_s^2/\mu\Delta$? There are a host of possibilities. As $\mu^{(BS)}$ is turned on, the energy of baryons (quasi-particles) carrying anti-strange quarks gets lowered, until eventually on has massless modes ("gapless superconductivity") at nonzero quark masses. This can be seen in the chiral Lagrangian by including the baryons. But then new instabilities seem to arise in the baryon current. Perhaps these lead to formation of a spatially inhomogeneous condensate, crystalline superconductivity.

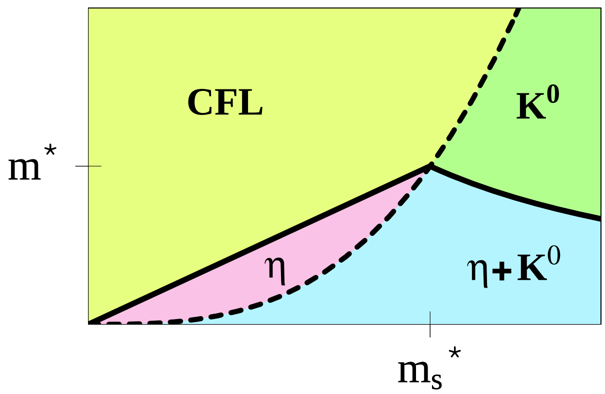

Figure 15: *The phase diagram as a function of light quark mass m and strange quark mass m_s and $m \equiv (m_u+m_d)/2$. The phases marked "CFL"," K^0", "η", and "$\eta+K^0$" are respectively the CFL phase without meson condensation, with kaon condensation, with η/η' condensation, and with both η/η' and K^0 condensation. Phase transitions are represented by a solid line if first order, a dashed line if second order. The location of the tetracritical point (m^*, m_s^*) is parametrically $m^* = O(\Delta\alpha_s^{3/4})$, $m_s^* = O(\Delta\alpha_s^{1/4})$; they are calculated in ref. [46].*

We see that even at very high densities where QCD is weakly coupled, dense quark matter exhibits an incredibly rich phase structure, still far from fully understood. Effective field theory has played an important role in unravelling this structure. We can only hope that some day numerical techniques will be available to explore this phase structure directly from QCD in regimes where QCD is strongly coupled, and where EFT techniques fail.

Problems:

V.1) The Lorentz group is equivalent to a complexified version of $SU(2) \times SU(2)$ and irreducible representations can be labeled as (j_1, j_2) where the j's are half-integers. Left-handed 2-component Weyl fermions transform as $(\frac{1}{2}, 0)$ and right-handed as $(0, \frac{1}{2})$. Instead of using 4-component Dirac spinors, a quark can be represented by two left-handed Weyl fields q which annihilates left-handed quarks, and q^c, which annihilates left-handed anti-quarks. This is more convenient than Dirac spinors when baryon number is not conserved. Each of these comes in three flavors and three colors, with the quantum numbers

	Lorentz	$SU(3)_L$	$SU(3)_R$	$SU(3)_c$
q	$(\frac{1}{2}, 0)$	3	1	3
q^c	$(0, \frac{1}{2})$	1	$\bar{3}$	$\bar{3}$

(174)

a) Show that there could not be Lorentz invariant $\langle qq^c \rangle$ condensate.

b) Show that could be a Lorentz invariant $\langle qq \rangle$ condensate, which would transform in the reducible $(\bar{3}, 1, \bar{3}) \oplus (6, 1, 6)$ representation of $SU(3)_L \times SU(3)_R \times SU(3)_c$. (Don't forget that fermion fields anticommute).

c) Show that if the condensate $\langle q_{\alpha ia} q_{\beta jb} \rangle \propto \epsilon_{\alpha\beta} \epsilon_{abx} \epsilon_{ijx}$ (where $\alpha, \beta = 1, 2$ are Lorentz indices, $i, j = 1, 2, 3$ are $SU(3)_L$ flavor indices, $a, b = 1, 2, 3$ are color indices, and the index x is summed over) corresponds to a Lorentz singlet condensate in the attractive color $\bar{3}$ channel. Show that it breaks $SU(3)_L \times SU(3)_c$ down to a diagonal $SU(3)$.

d) Consider the analogous $\langle (q^c)^{ia}_{\dot{\alpha}} (q^c)^{jb}_{\dot{\beta}} \rangle \propto \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon^{abx} \epsilon^{ijx}$ (where $\dot{\alpha}, \dot{\beta} = 1, 2$ are Lorentz indices, $i, j = 1, 2, 3$ are $SU(3)_R$ flavor indices, $a, b = 1, 2, 3$ are color indices) breaks $SU(3)_R \times SU(3)_c$ down to a diagonal $SU(3)$.

e) Taken together, show that $SU(3)_L \times SU(3)_R \times SU(3)_c$ is broken down to a diagonal $SU(3)$, breaking 16 symmetry generators. What happens to the sixteen Goldstone bosons?

V.2) In eq. (162) I gave a gauge invariant order parameter for the breaking of $SU(3) \times SU(3) \times U(1)_A$ symmetry. Find a gauge invariant parameter for the breaking of baryon number. What discrete subgroup of baryon number symmetry is left unbroken? Can you see this symmetry in the gauge variant formulation of eq. (161)? (Hint: find all $U(1)_B$ transformations of the quark bilinear which can be undone by $SU(3)_c$ transformations).

V.4) Expand the QCD Lagrangian in terms of $\psi_\pm$ and Derive eq. (166).

V.5) Start with the QCD Lagrangian expanded in terms of $\psi_\pm$. Consider vacuum energy diagrams with a single quark loop, arbitrary insertions of the mass matrix M and $M^\dagger$, arbitrary insertions of the quark condensate $\langle q^a_{L,i} q^b_{L,j} \rangle \propto \epsilon^{abx} \epsilon_{ijx}$ and an arbitrary dressing of gluons.

a) By following chirality around the loop, show that any mass dependence of the vacuum energy from such diagrams involves even powers of M and/or $M^\dagger$, and that therefore there is no contribution to the vacuum energy at $O(M)$. (Recall that M couples $\psi_L^\dagger$ to ψ_R or $\psi_R^\dagger to \psi_L$; the gluon couples $\psi_L^\dagger$ to ψ_L or $\psi_R^\dagger$ to ψ_R; and that the condensate couples ψ_L to ψ_L or ψ_R to ψ_R (and the conjugate couplings). Draw the analogous diagrams for QCD at $\mu = 0$, including insertions of the $\bar{q}q$ condensate, and show why there *is* and $O(M)$ contribution in this case, reflected by the $\mathrm{Tr}M\Sigma$ term in the chiral Lagrangian, eq. (69).

b) Find a one fermion loop vacuum energy contribution with two insertions of M on the fermion line and insertions of the condensate as needed to make the diagram not vanish. By considering flavor flowing around the diagram, paying attention to the ϵ's in the condensate insertions, show that this vacuum energy contribution is proportional to $\epsilon_{ijx}\epsilon^{rsx}M_r^iM_s^j$. Note that this vanishes if any two quark masses vanish. Such diagrams give rise to the $O(M^2)$ terms in the CFL chiral Lagrangian, eq. (168).

6 Epilog

Effective field theory is a tool, and writing a comprehensive review of its development and applications would be like writing a treatise on the hammer and its applications. Instead I have tried to give you a general idea how effective field theory works and can be used, citing papers I find interesting or pedagogical, rather than trying to be historical, often focusing on ways that I have used EFT in my own research.

A Partial answers to selected problems

III.2)

$$P: \quad \Sigma \to \Sigma^\dagger \;, \qquad C: \quad \Sigma \to \Sigma^T \;. \tag{175}$$

The only mesons which are eigenstates under CP are the neutral ones; π^0 and η; they are odd under P, even under C and therefore odd under CP.

III.3) The vacuum energy arising from $\mathcal{L}_M$ is $-f^2\mathrm{Tr}(\widetilde{\Lambda}M)\Sigma + \mathrm{h.c.}$ where Σ is the constant $SU(3)$ matrix characterizing the alignment of the quark condensate. Assuming the quark masses to be positive, for positive $\widetilde{\Lambda}$, this term is minimized for Σ being the unit matrix, which leaves Gell-Mann's $SU(3)_V$ unbroken. For negative $\widetilde{\Lambda}$, however, $\Sigma = \mathrm{diag}(1, -1, -1)$ is the $SU(3)$ matrix which minimizes the energy. This corresponds to a ground state which spontaneously breaks isospin which does not look like our world.

III.4) To preclude mixing between the axion and the π^0 and η choose $X = M^{-1}/\mathrm{Tr}M^{-1}$. Dropping terms of $O(m_d/m_s)$, one finds that the axion mass $m_a = (f_\pi m_\pi/f_a) \times \sqrt{Z}/(1+Z)$, where $Z \equiv (m_u/m_d) \simeq 1/2$.

III.6) The photon couples to $\bar{q}_L\gamma^\mu Q_L q_L + \bar{q}_R\gamma^\mu Q_R q_R$, where $Q_L = Q_R = \mathrm{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ are $SU(3)$ matrices. $Q_{L,R}$ can be treated as spurions, transforming as octets: $Q_L \to LQ_L L^\dagger$, $Q_R \to RQ_R R^\dagger$. An invariant operator is then $\mathrm{Tr} Q_L \Sigma Q_R \Sigma^\dagger$, which you can show gives the charged pions and kaons an equal mass shift.

Note that in dimensional regularization and Landau gauge, the one-loop electromagnetic corrections arising from the kinetic term vanish.

III.7) Yes. First consider the definition of f_π in eq. (63). The current is summed over N_c quark colors and so consists of a sum of N_c terms. From basic quantum mechanics this means that to normalize the current so that it created a meson with probability $O(1)$ in the large N_c limit would require multiplying the current by $1/\sqrt{N_c}$. From eq. (63) it therefore follows that $f_\pi \propto \sqrt{N_c}$ for large N_c. On the other hand, you should think of Λ as being a typical vector meson mass, set by the intrinsic QCD scale Λ_{QCD}. The large N_c limit is taken in QCD by rescaling the coupling g keeping $N_c g^2$ fixed, which leaves Λ_{QCD} unchanged, and so we expect in the chiral Lagrangian $\Lambda = O(1)$ for large N_c, and $4\pi f_\pi/\Lambda = O(\sqrt{N_c})$. It follows that radiative corrections are negligible compared to higher order tree-level terms.

B Dimensional regularization

For your convenience, I am including the standard formulas used in dimensional regularization.

B.1 Useful integrals as a function of dimension

Consider the following integral in d dimensions with a *Euclidian* metric:

$$I_1 \equiv \int \mathrm{d}^d k \; \frac{1}{(k^2+a^2)^r} \; . \tag{176}$$

We may evaluate this making in terms of the Γ function:

$$\alpha^{-s}\Gamma(s) = \int_0^\infty \mathrm{d}x \; x^{s-1} \; \mathrm{e}^{-\alpha x}. \tag{177}$$

Then

$$\begin{aligned} I_1 &= \frac{1}{\Gamma(r)} \int \mathrm{d}^d k \int_0^\infty \mathrm{d}x \; x^{r-1} \; \mathrm{e}^{-x(k^2+a^2)} \\ &= \frac{\pi^{d/2}}{\Gamma(r)} \int_0^\infty \mathrm{d}x \; x^{r-1-d/2} \; \mathrm{e}^{-xa^2} \\ &= \pi^{d/2} a^{d-2r} \frac{\Gamma(r-d/2)}{\Gamma(r)} \end{aligned} \tag{178}$$

Another useful integral is

$$I_2 \equiv \int \mathrm{d}^d k \; \frac{k^2}{(k^2+a^2)^r} \; . \tag{179}$$

To get this we define

$$\begin{aligned} I_1(\alpha) &\equiv \int \mathrm{d}^d k \; \frac{1}{(\alpha k^2+a^2)^r} \\ &= \alpha^{-d/2} I_1 \; ; \end{aligned} \tag{180}$$

then by differentiating by α and setting $\alpha = 1$ we find

$$I_2 = \frac{d\pi^{d/2} a^{d-2r+2}}{2(r-1)} \frac{\Gamma(r-1-d/2)}{\Gamma(r-1)} \; . \tag{181}$$

Finally note that

$$\begin{aligned} I_3^{\mu\nu} &\equiv \int \mathrm{d}^d k \; \frac{k^\mu k^\nu}{(k^2+a^2)^r} \\ &= \frac{\delta^{\mu\nu}}{d} I_2 \; . \end{aligned} \tag{182}$$

B.2 Some properties of the Γ function

Gamma functions have the property $\Gamma(z+1) = z\Gamma(z)$, with $\Gamma(1) = 1$. Thus for integers $n \geq 0$,

$$\Gamma(n+1) = n!, \qquad n \geq 0 . \tag{183}$$

Also useful is the value

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} . \tag{184}$$

The Gamma function is singular for non-positive integer arguments. Near these singularities it can be expanded as

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n}\left[\frac{1}{\epsilon} + \psi(n+1) + \mathcal{O}(\epsilon)\right] , \tag{185}$$

where

$$\begin{aligned} \psi(n+1) &= 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma , \\ \gamma &= 0.5772\ldots \end{aligned} \tag{186}$$

In particular,

$$\begin{aligned} \Gamma(\epsilon-1) &= -\frac{1}{\epsilon} + \gamma - 1 \\ \Gamma(\epsilon) &= \frac{1}{\epsilon} - \gamma \end{aligned} \tag{187}$$

B.3 Common integrals in $d \to 4$ dimensions

It follows that two of the most useful integrals are given by

$$\mu^{2\epsilon}\int \frac{\mathrm{d}^{4-2\epsilon}q}{(2\pi)^{4-2\epsilon}}\frac{1}{q^2+m^2} = \frac{m^2}{16\pi^2}\left[-\frac{1}{\epsilon} + \gamma - 1 - \ln 4\pi + \ln(m^2/\mu^2) + O(\epsilon)\right] \tag{188}$$

$$\mu^{2\epsilon}\int \frac{\mathrm{d}^{4-2\epsilon}q}{(2\pi)^{4-2\epsilon}}\frac{1}{(q^2+m^2)^2} = \frac{1}{16\pi^2}\left[\frac{1}{\epsilon} - \gamma + \ln 4\pi - \ln(m^2/\mu^2) + O(\epsilon)\right] . \tag{189}$$

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